Palm Calculus
Bonus

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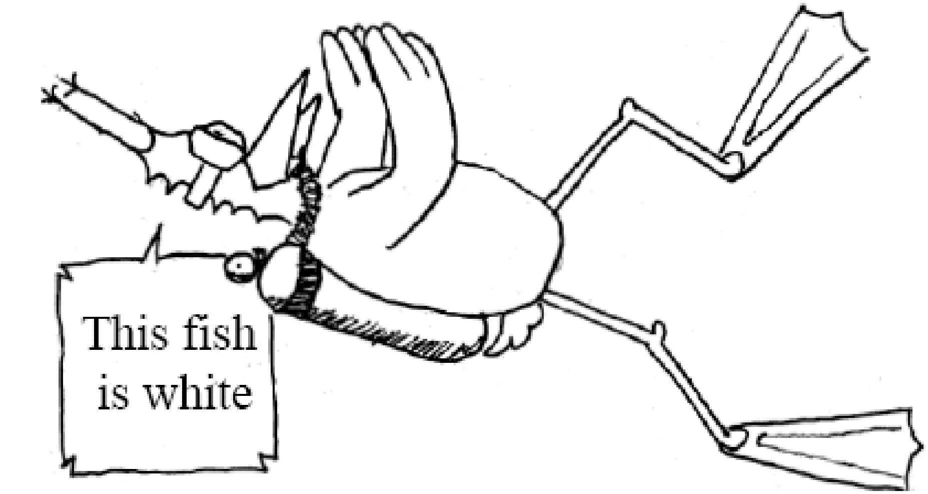
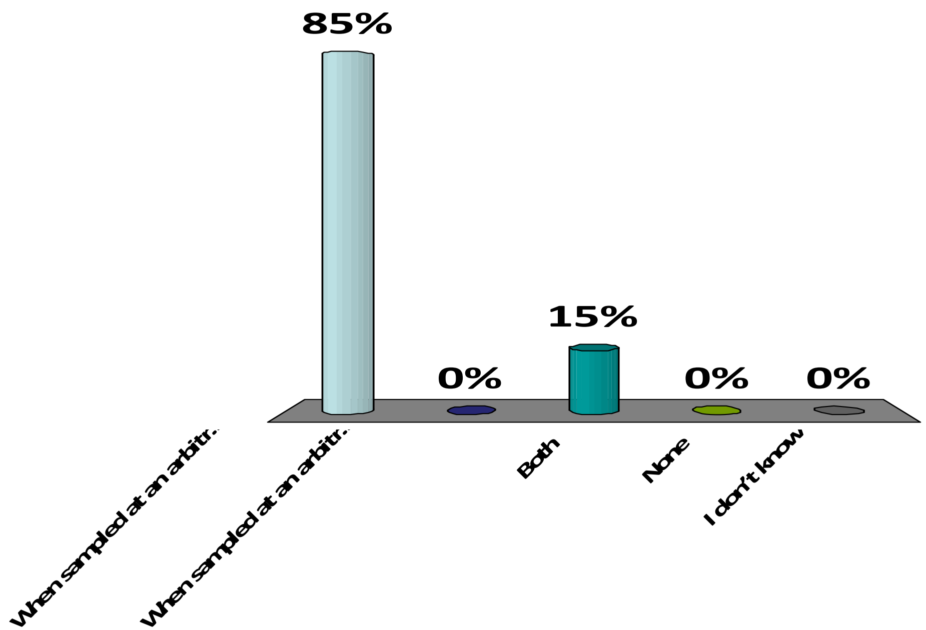


For the random waypoint model, the distribution of the next waypoint is uniform…

A. When sampled at an arbitrary waypoint
B. When sampled at an arbitrary point in time
C. Both
D. None
E. I don't know



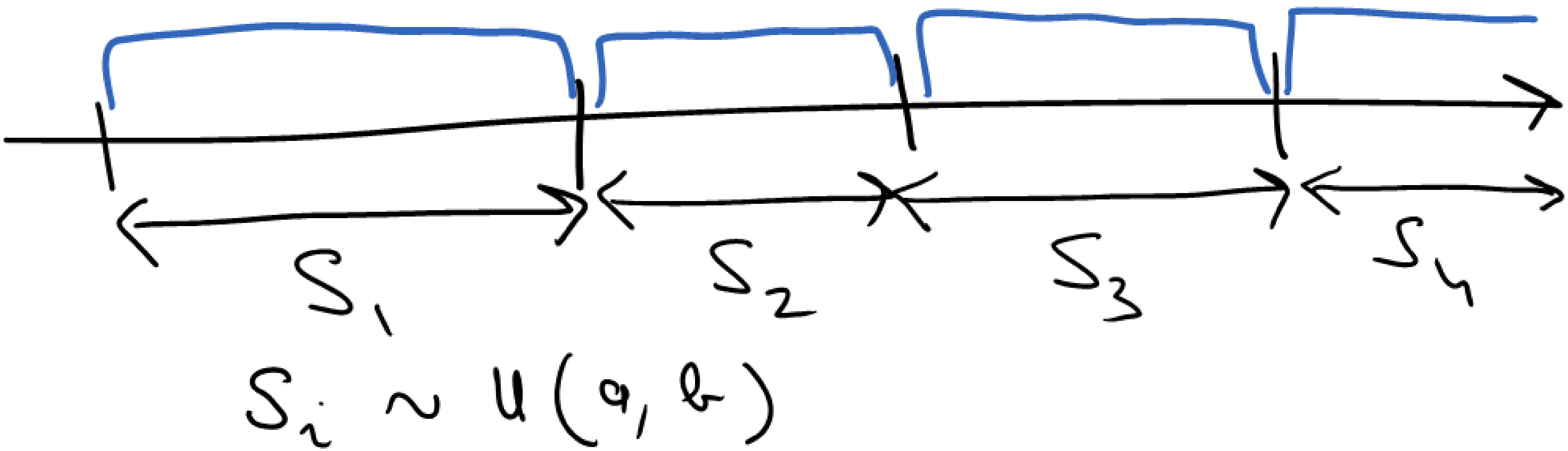
Solution

A is true by definition of the model

B is not true, we sample more often trips that are long; long trips are more likely to have ends at the edge of the area. The next waypoint, seen at an arbitrary point in time, is more often at the edge than at the center

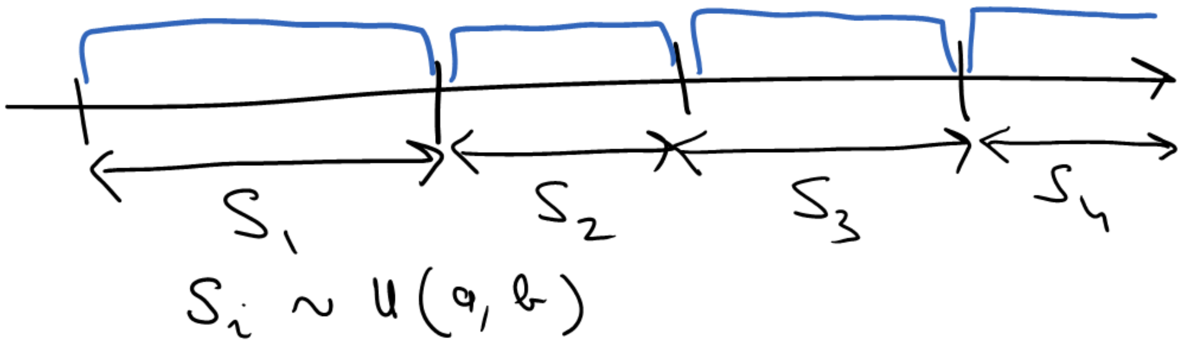
Devices run in a cycle of duration $\sim U\ (a, b)$

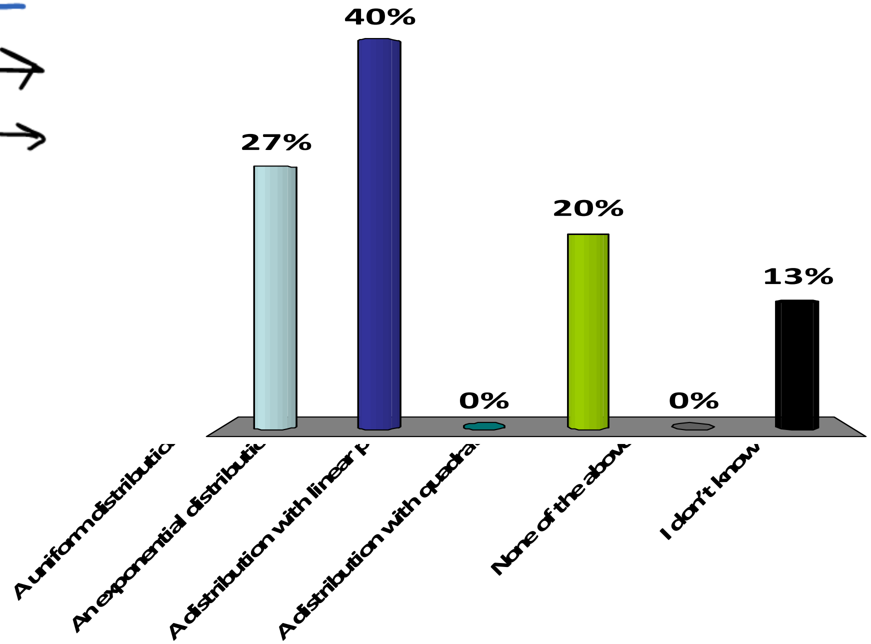
We simulate a number of devices. We want to avoid removing transients and start the simulation by sampling the residual times . From which distribution should we sample the residual cycle times ?



From which distribution should we sample the residual cycle time ?

A. A uniform distribution
B. An exponential distribution
C. A distribution with linear pdf
D. A distribution with quadratic pdf
E. None of the above
F. I don't know





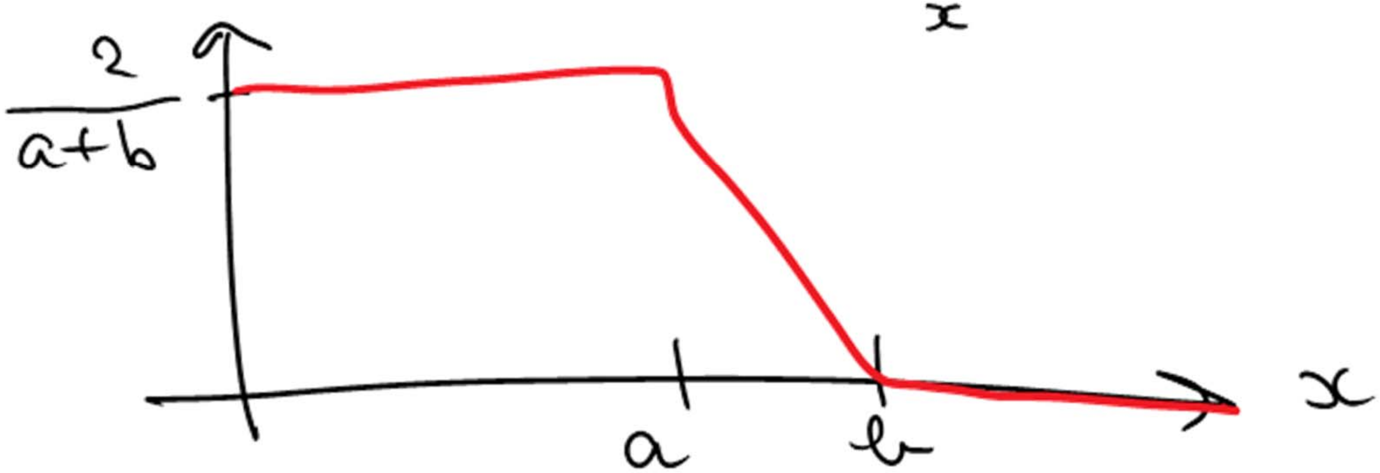
Solution

Residual Time: $f_X(x) = \lambda \int_x^{+\infty} f_T(t)\,dt$

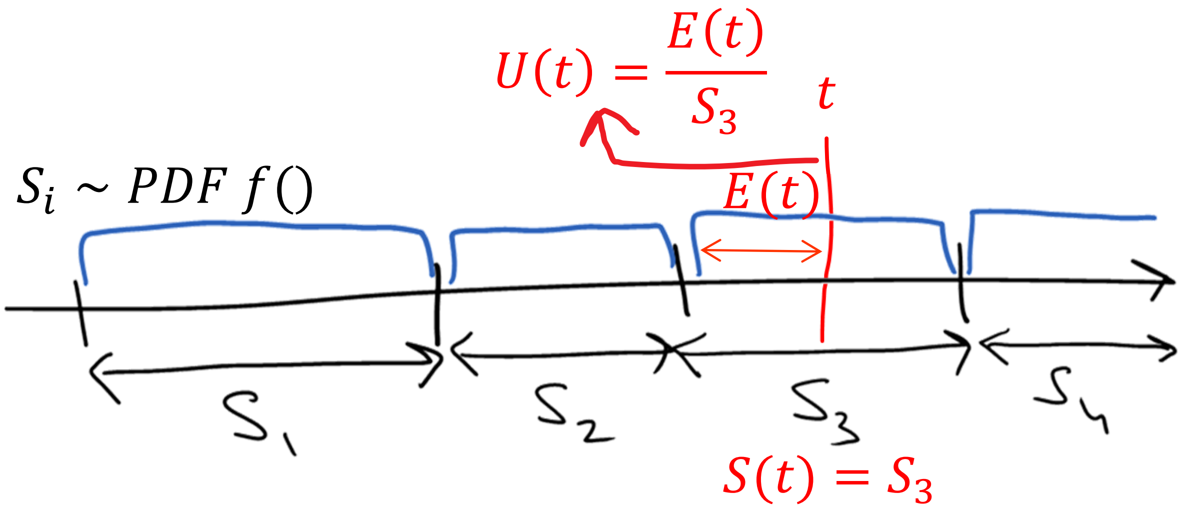
$f_T(t) = \frac{1}{b-a} 1_{b \le t \le a}$; $\lambda^{-1} = \text{mean} \Rightarrow \lambda = \frac{2}{a+b}$

Let $0 \le x \le b$ (else $f_X(x) = 0$)

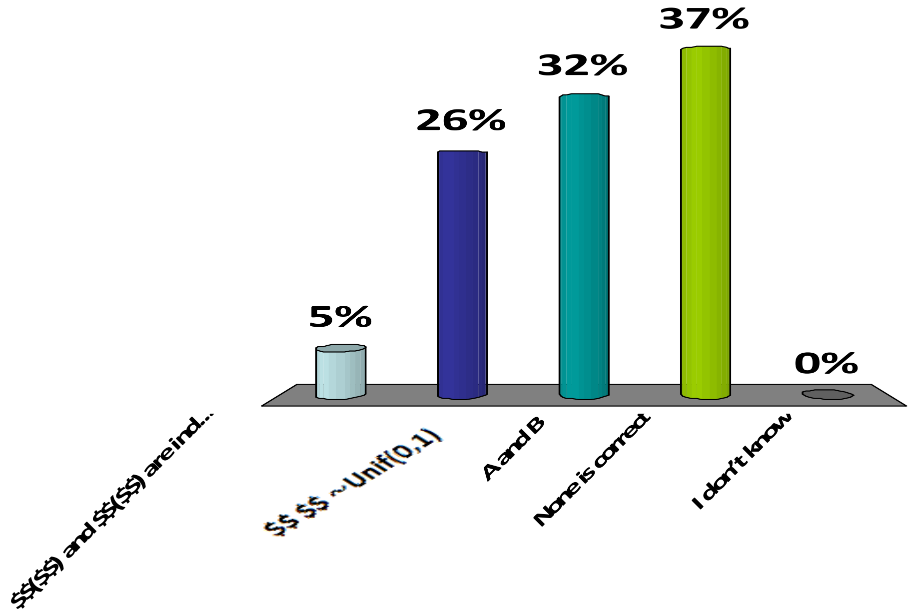
- if $0 \le x \le a$: $f_X(x) = \lambda \int^b f_T(t)\,dt = \lambda \int_a^b f_T(t)\,dt = \lambda$
- if $a \le x \le b$: $f_X(x) = \lambda \int_x^b \frac{dt}{b-a} = \lambda \frac{b-x}{b-a}$



$U(t)$ is the fraction of the current interval that has elapsed at time t.
The current interval is $S(t)$



A. $U(t)$ and $S(t)$ are independent
B. $U(t) \sim$ Unif(0,1)
C. A and B
D. None is correct
E. I don't know



Solution

We want the joint distribution of $S(t)$ and $U(t)$ so we compute $E\left(\varphi\big(S(t)\big)\psi(U(t))\right)$ for any test functions φ, ψ. By Palm's inversion formula:

$$E\left(\varphi\big(S(t)\big)\psi(U(t))\right) = \lambda E^0\left(\int_0^{T_1} \varphi\big(S(s)\big)\psi\big(U(s)\big)ds\right) =$$

$$\lambda E^0\left(\int_0^{T_1} \varphi(T_1)\psi\left(\frac{s}{T_1}\right)ds\right) = \lambda E^0\left(\varphi(T_1)\int_0^{T_1} \psi\left(\frac{s}{T_1}\right)ds\right)$$

By the change of variable in the integral $s = T_1 u$:

$$= \lambda E^0\left(\varphi(T_1)\int_0^1 \psi(u)T_1 du\right) = \lambda E^0\left(\varphi(T_1)T_1\int_0^1 \psi(u)du\right)$$

$$= \lambda\left(\int_0^1 \psi(u)du\right)E^0(\varphi(T_1)T_1) = \int_0^1 \psi(u)du \int_0^\infty \varphi(s)\lambda s f(s)ds$$

$$= \int_{[0,\infty[\times[0,1]} \varphi(s)\psi(u)\lambda s f(s)ds\, du$$

Solution

Thus the joint pdf is

$$f_{S(t),U(t)}(s,u) = \lambda s f(s) 1_{u\in[0,1]}$$

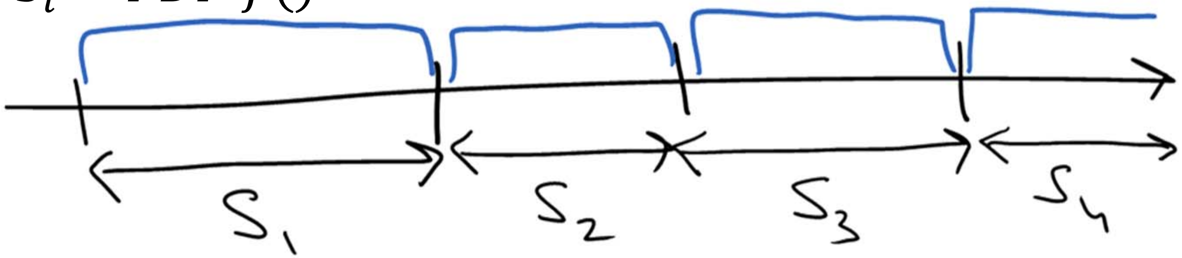
which shows that

1. $S(t)$ and $U(t)$ are independent
2. the pdf of $S(t)$ is $\lambda s f(s)$ (we knew this already)
3. The pdf of $U(t)$ is $1_{u\in[0,1]}$ i.e $U(t)$ is uniform in [0,1]

Answer C

An algorithm for the perfect simulation of a device is returning the duration of the current interval S and the residual time R until end of current interval.

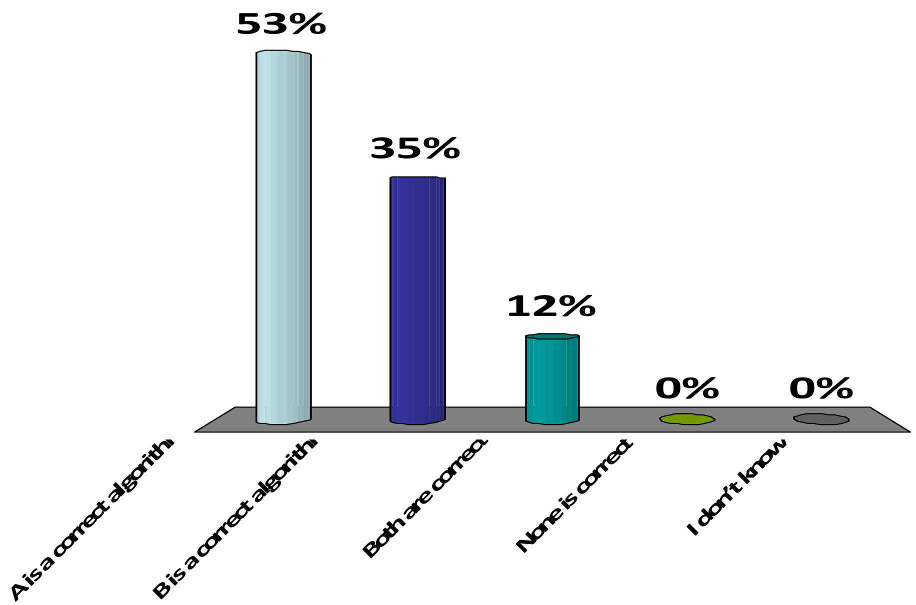
$S_i \sim PDF\ f()$



A
1. Draw S from distribution with pdf $g(s) = Ksf(s)$ where K is a constant
2. Draw $U \sim$ Unif(0,1)
3. $R = US$

B
1. Draw S from distribution with pdf $f(s)$
2. Draw $U \sim$ Unif(0,1)
3. $R = US$

A. A is a correct algorithm
B. B is a correct algorithm
C. Both are correct
D. None is correct
E. I don't know



Solution

S should be drawn from the pdf of the interval duration seen by at an arbitrary point in time, which is not equal to $f(s)$, hence B is wrong.

To draw R and S we can draw U and S and note that $R = S(1-U)$. Therefore a correct algorithm is
1. Draw S from distribution with pdf $g(s) = Ksf(s)$ where K is a constant
2. Draw $U \sim \text{Unif}(0,1)$
3. $R = (1-U)S$

But since $(1-U) \sim \text{Unif}(0,1)$ we can replace lines 2,3 by
2. Draw $V \sim \text{Unif}(0,1)$
3. $R = VS$

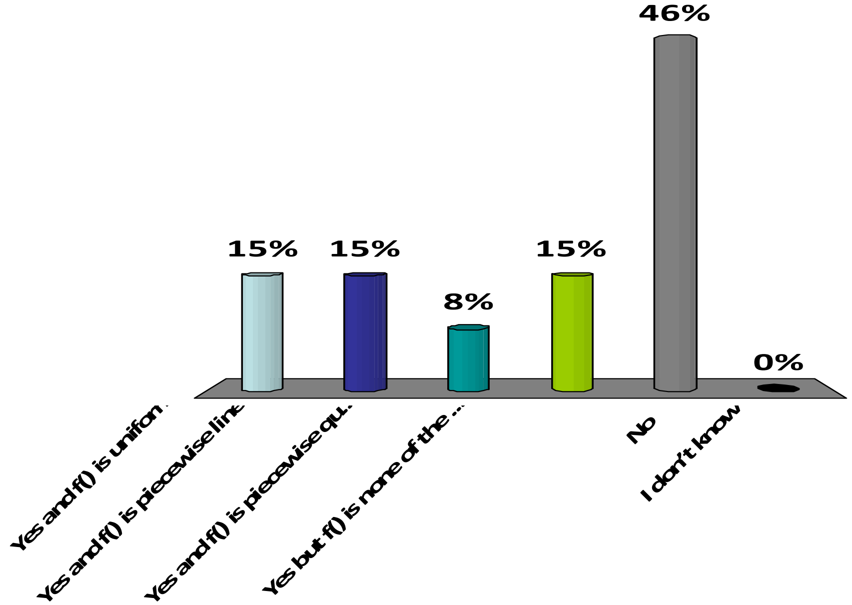
which is the same as algorithm A. Answer A.

Consider the random waypoint model, where the speed chosen at a waypoint is sampled from the pdf f(). Can we choose f() such that

1. the model has a stationary regime and

2. the distribution of speed sampled at an arbitrary point in time is uniform ?

A. Yes and f() is uniform
B. Yes and f() is piecewise linear
C. Yes and f() is piecewise quadratic
D. Yes but f() is none of the above
E. No
F. I don’t know

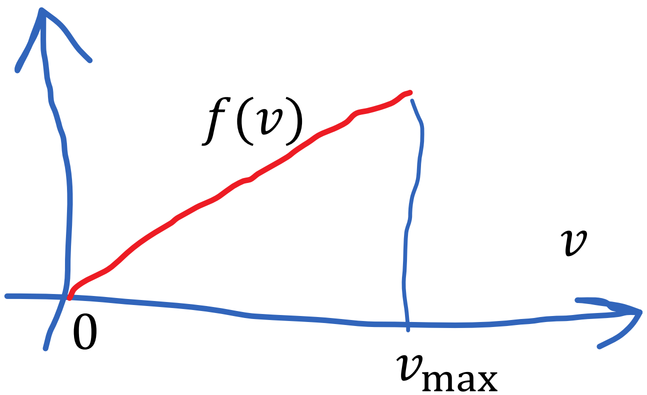


Solution

Inversion formula $\Rightarrow$ $f_{V(t)}(v) = \frac{K}{v} f(v)$ for some K

$f_{V(t)}$uniform $\Rightarrow f_{V(t)} = \frac{1}{v_{\max}} 1_{0 \le v \le v_{\max}}$

$$\Rightarrow f(v) = \frac{v}{K v_{\max}} 1_{0 \le v \le v_{\max}}$$



$E^0\left(\frac{1}{V_0}\right) = \int_0^{v_{\max}} \frac{1}{v} \frac{v}{K v_{\max}} dv < \infty$ therefore the average trip duration is finite and there is a stationary regime – Answer B

A wireless channel has a fluctuating rate $r(t)$. A system sends data over this channel in rounds, of average duration $\bar{T}$. The average amount of data transferred in one round is $\bar{B}$.

We sample the channel using a Poisson process of rate λ and find a sampled rate $\bar{r}$

A. $\bar{r} = \frac{\bar{T}}{\bar{B}}$

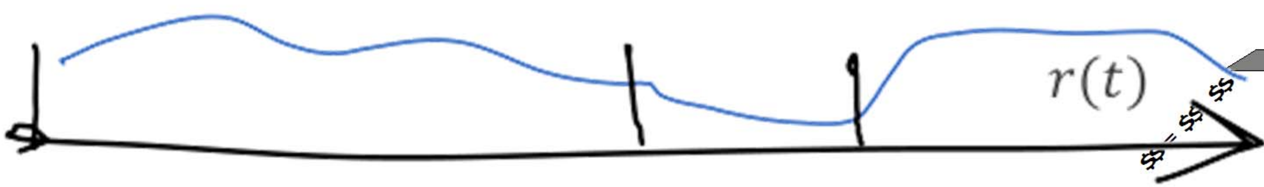
B. $\bar{r} = \frac{\bar{B}}{\bar{T}}$

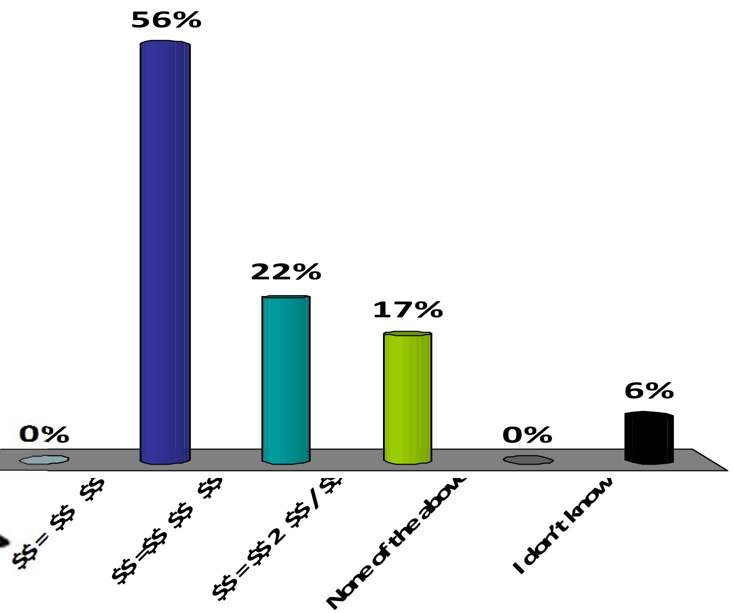
C. $\bar{r} = \lambda \frac{\bar{B}}{\bar{T}}$

D. $\bar{r} = \lambda^2 \bar{T} \bar{B}$

E. None of the above

F. I don't know

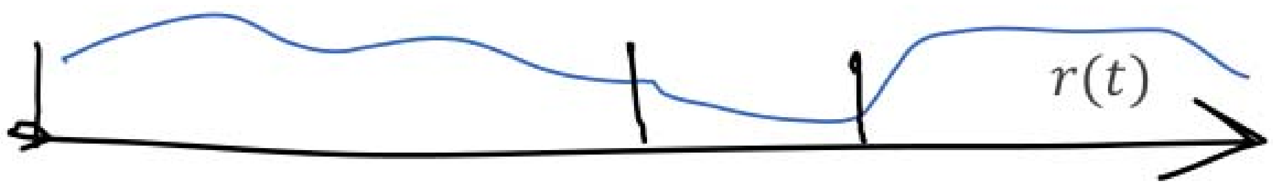




Solution

A and C are not correct because the units don’t match

Answer B is correct; we give two proofs



PASTA $\Rightarrow \bar{r} = \mathbb{E}(r(t))$

1) Large Time Heuristic: TTOT $\quad \bar{r} = \frac{1}{TTOT}\int_0^{TTOT} r(s)\,ds$

$= \frac{1}{TTOT}$ Amount of data transmitted $= \frac{1}{TTOT} NTOT\,\bar{B} = \frac{\bar{B}}{\bar{T}}$

NTOT Rounds

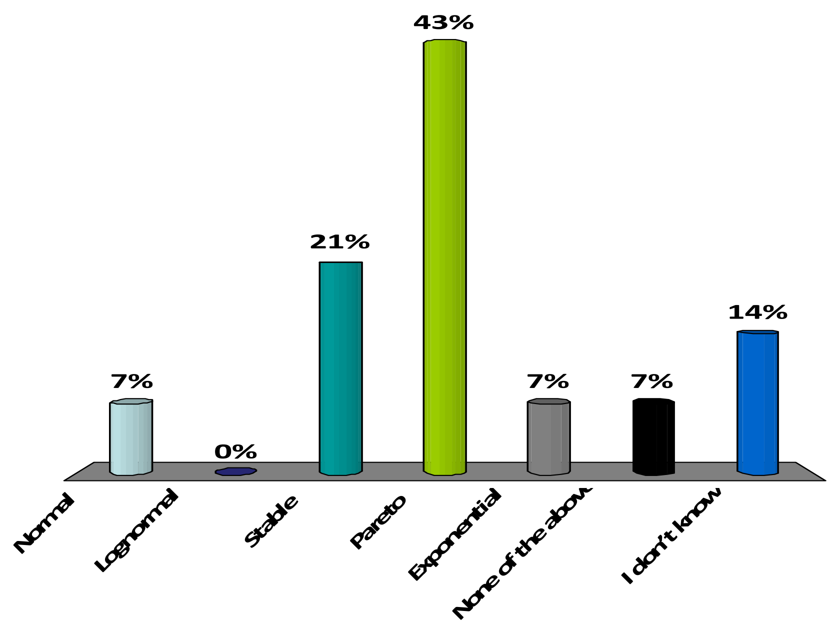
2) $\mathbb{E}(r(t)) = \lambda_1 \mathbb{E}^0\left(\int_{T_0}^{T_1} r(s)\,ds\right) = \frac{1}{\bar{T}}\bar{B}$

We measure the distribution of flow sizes, in packets, transferred from a server. We find standard Pareto(p) with $p > 1$. i.e. with PDF

$$f(x) = \frac{p}{x^{p+1}} 1_{x \geq 1}$$

What is the pdf of the size of a flow seen by an arbitrary packet?

A. Normal
B. Lognormal
C. Stable
D. Pareto
E. Exponential
F. None of the above
G. I don't know



Solution

$$f_P(x) = \lambda x f_F(x)$$

$$f_F(x) = \frac{p}{x^{p+1}} 1_{x \geq 1}$$

$$\Rightarrow f_P(x) = \frac{\lambda p}{x^{p+1}} \cdot x = \frac{\lambda p}{x^p} 1_{x \geq 1}$$

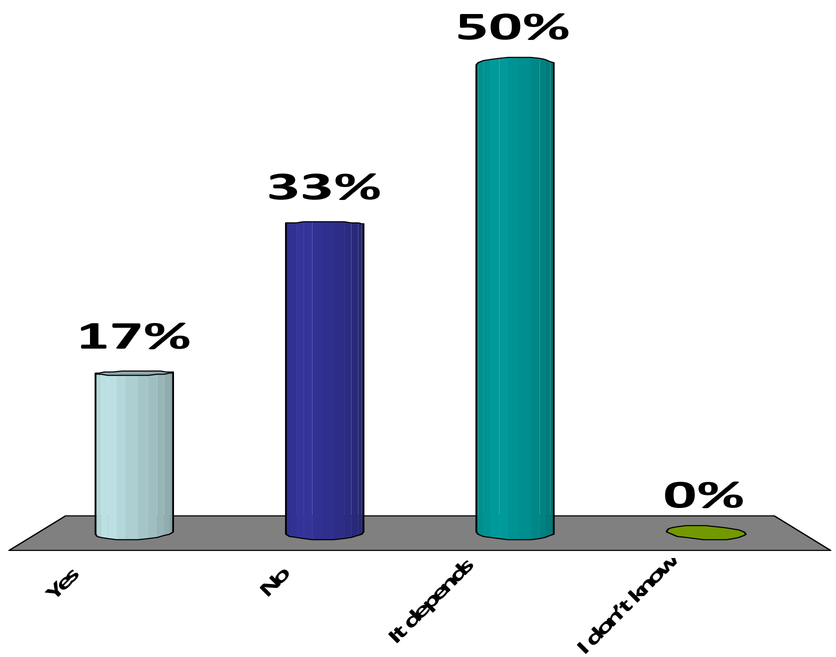
$\lambda p = p - 1$ necessarily, ie $\lambda = 1 - \frac{1}{p}$

Flow sizes seen by packets is Pareto $(p-1)$

The distribution of flow sizes in packet, seen by packets, is heavy tailed.

Therefore the distribution of flow sizes is heavy tailed.

A. Yes
B. No
C. It depends
D. I don't know

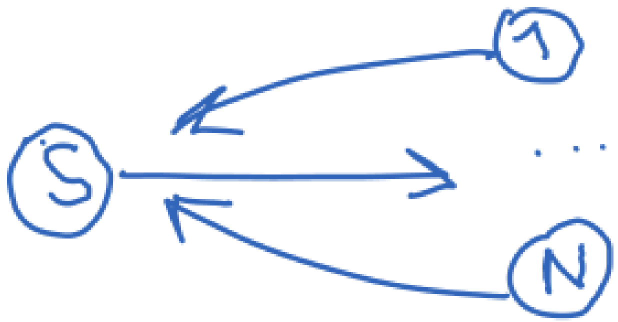


Solution

Consider the previous example
Seen by packets: Pareto(p-1); heavy tailed if $p < 3$
Seen by flows: Pareto(p): heavy tailed if $p < 2$

When $2 < p < 3$ distribution seen by flows is not heavy tailed
When $p < 2$ distribution seen by flows is ~~not~~ heavy tailed

Answer C



A server sends a broadcast poll and waits for all clients to ACK.

The round trip times $S \to i \to S$ for clients i are iid $\sim \mathrm{Exp}(1)$

How many polls per time unit are sent ?

A. $\lambda \approx \frac{1}{\log N}$

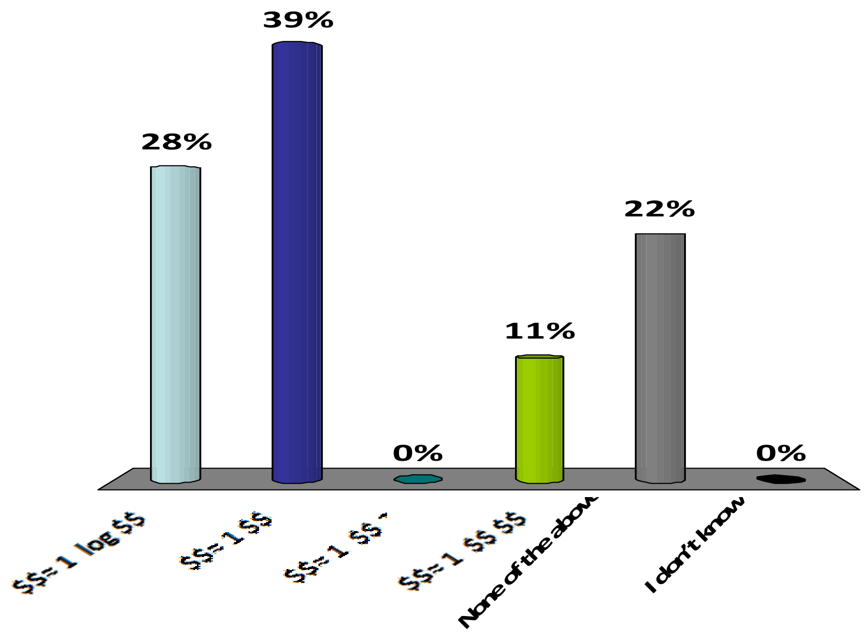
B. $\lambda \approx \frac{1}{N}$

C. $\lambda \approx \frac{1}{N^2}$

D. $\lambda \approx \frac{1}{N^p}$

E. None of the above

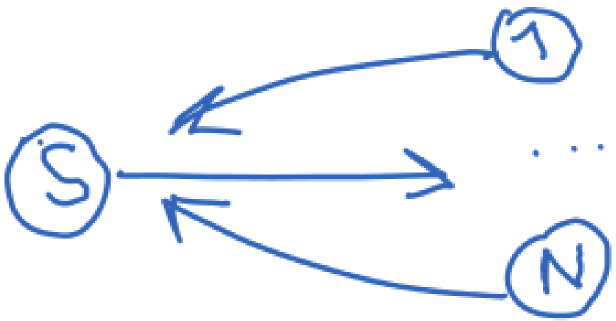
F. I don't know



Solution

1. The answer is the intensity of the point process of polls.
$$\lambda = \frac{1}{E^0(poll\ time)}$$
and the expectation is always finite
2. The expected poll time is
$$T = \max(X_1, \ldots, X_N)$$
Thus (see qqplots) $E(T) \approx F^{-1}\left(\frac{N}{N+1}\right) = \log(N+1)$
3. $\lambda \approx \frac{1}{\log(N+1)} \approx \frac{1}{\log N}$

Answer A



A server sends a broadcast poll and waits for all clients to ACK.

The round trip times $S \to i \to S$ for clients i are iid $\sim \mathrm{Pareto}(p)$ with $p > 1$. How many polls per time unit are sent ?

A. $\lambda \approx \frac{1}{\log N}$

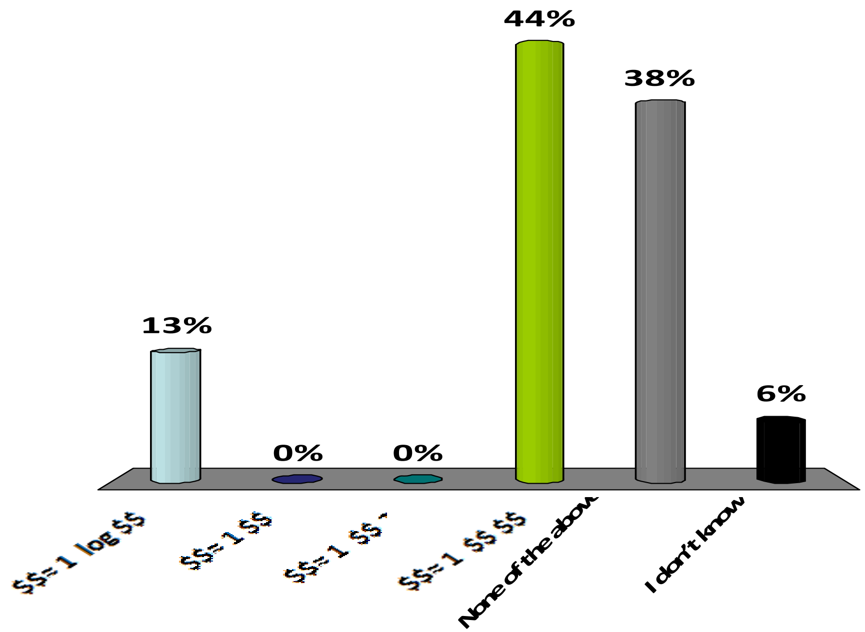
B. $\lambda \approx \frac{1}{N}$

C. $\lambda \approx \frac{1}{N^2}$

D. $\lambda \approx \frac{1}{N^p}$

E. None of the above

F. I don't know



Solution

1. The answer is the intensity of the point process of polls.
$$\lambda = \frac{1}{E^0(poll\ time)}$$
provided the expectation is finite, i.e. $p > 1$
2. The expected poll time is
$$T = \max(X_1, \dots, X_N)$$
Thus (see qqplots) $E(T) \approx F^{-1}\left(\frac{N}{N+1}\right) = (N+1)^{\frac{1}{p}}$
3. If $p > 1$ then $\lambda \approx \frac{1}{(N+1)^{\frac{1}{p}}}$

Answer E

Devices follow an ON/OFF cycle. The mean ON and OFF times are $\bar{S}$ and $\bar{T}$.

The probability that the device is ON at an arbitrary point in time is…

A. $\frac{\bar{S}}{\bar{T}}$

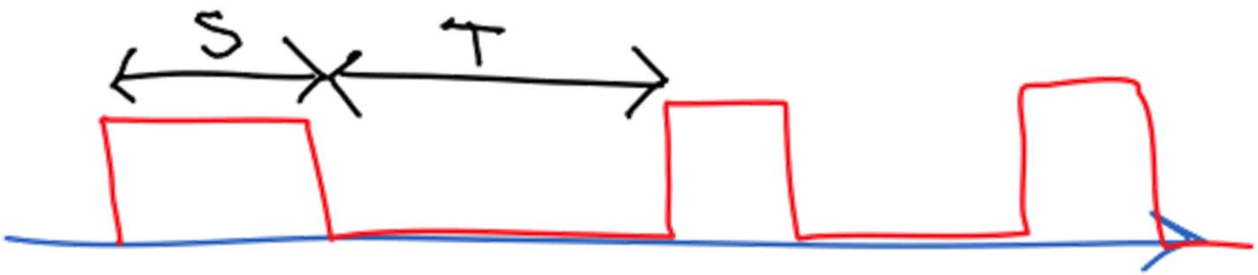
B. $\frac{\bar{S}\bar{T}}{(\bar{S}+\bar{T})^2}$

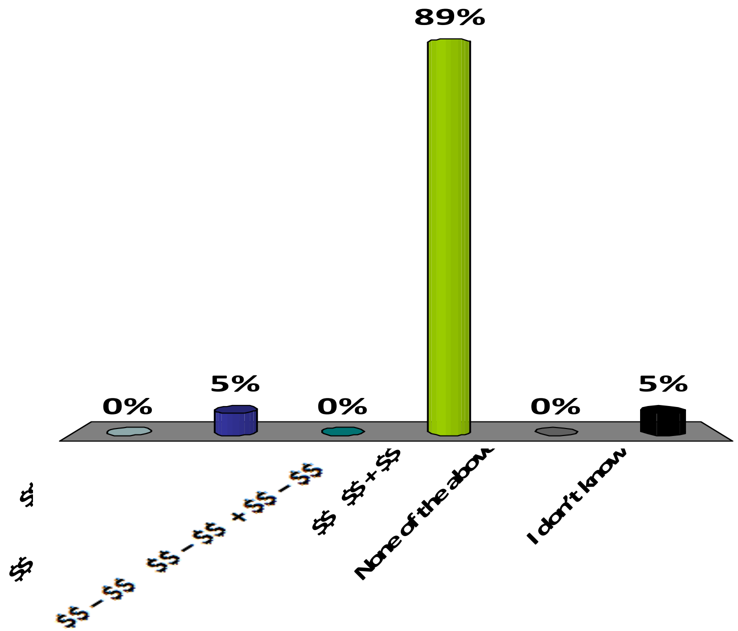
C. $\frac{e^{-\bar{S}}}{e^{-\bar{S}}+e^{-\bar{T}}}$

D. $\frac{\bar{S}}{\bar{S}+\bar{T}}$

E. None of the above

F. I don't know





Solution

Answer D

let $I(t)$ = ON or OFF

Consider point process of all transition instants

$$P(I(t) = ON) = \lambda E^0\left(\int_0^{T_1} 1_{I(s)=ON}\ ds\right)$$

$$= \lambda E^0(T_1 1_{this\ interval\ is\ ON})$$

Now $P^0(this\ interval\ is\ ON) = 0.5$

Thus $P(I(t) = ON) = \lambda \times 0.5 \times \bar{S}$

Idem $P(I(t) = OFF) = \lambda \times 0.5 \times \bar{T}$

Thus $\lambda \times 0.5 = \frac{1}{\bar{S} + \bar{T}}$

Thus $P(I(t) = ON) = \frac{\bar{S}}{\bar{S} + \bar{T}}$