Bonus Simulation

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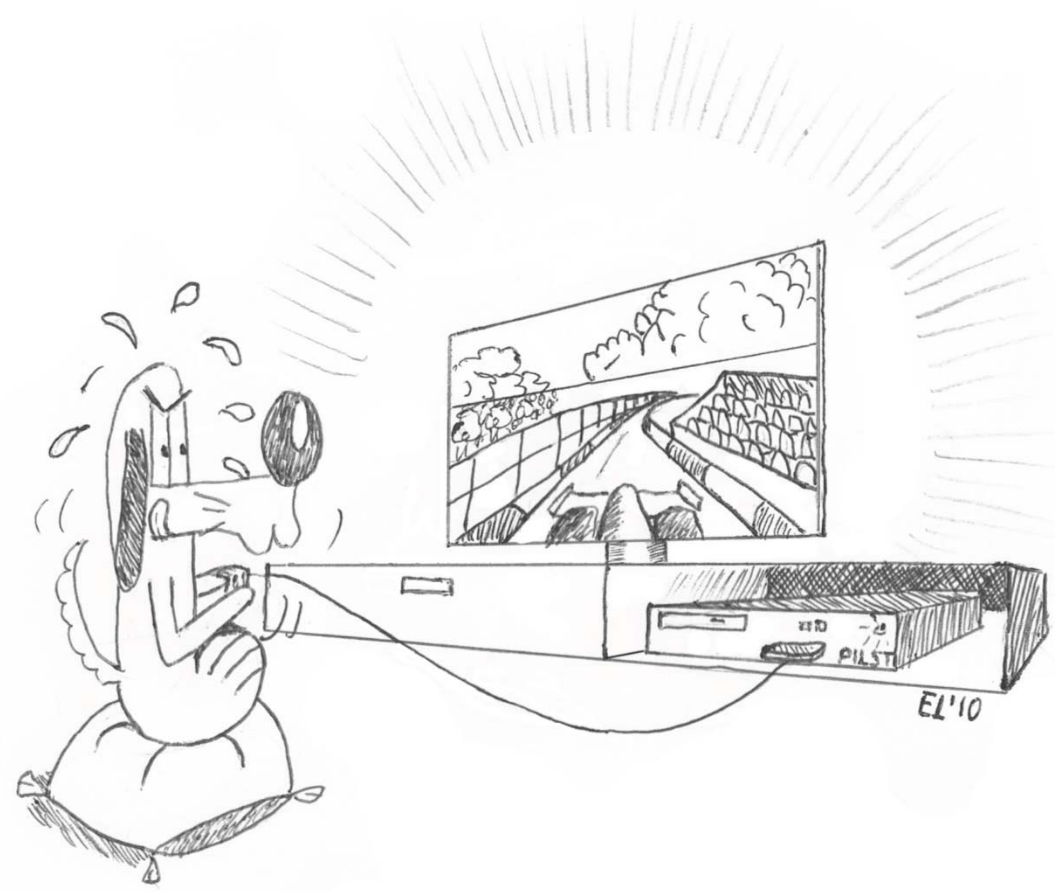
2015



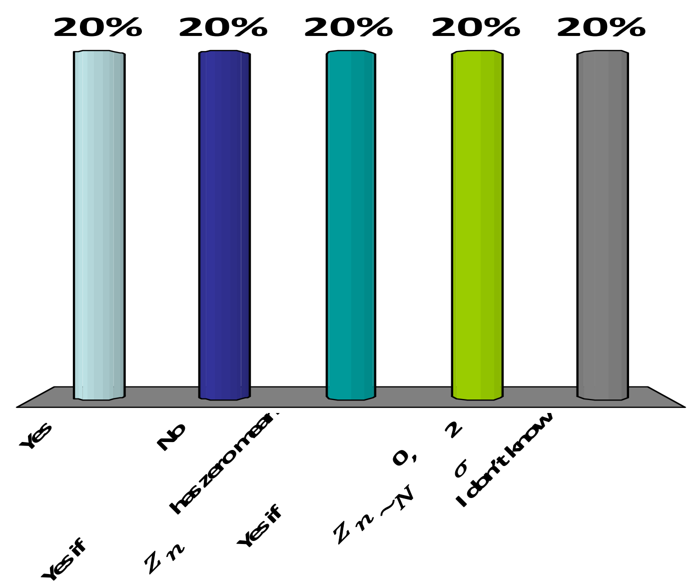
The sequence X_n is a random walk, i.e.

$$X_n = Z_1 + Z_2 + \cdots + Z_n$$

where Z_n is iid.

Is the sequence X_n stationary ?

A. Yes
B. No
C. Yes if Z_n has zero mean
D. Yes if $Z_n \sim N(0, \sigma^2)$
E. I don't know



Solution

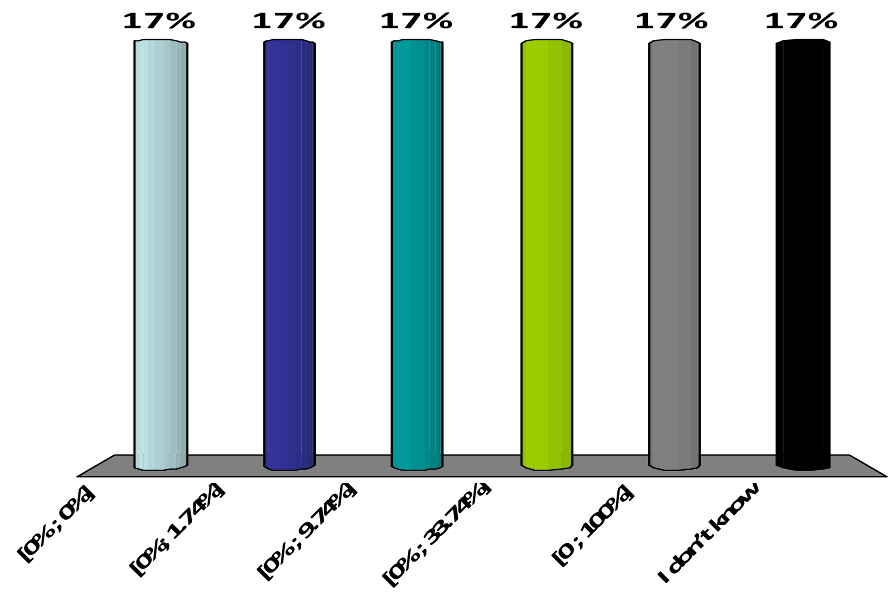
X_n is not stationary; assume for example Z_n has finite variance σ^2. The variance of X_n is $n\,\sigma^2$, it varies with n.

Answer B

Is the simulator output $(X_1, X_2, \ldots)$ stationary ?

$X_1 \sim N(\mu, \sigma^2)$
for $n = 2 : n_{\max}$
  $U \sim \text{Unif}(0,1)$
  if $U < p$
    $X_n = X_{n-1}$
  else
    $X_n \sim N(\mu, \sigma^2)$
end

A. Yes
B. No
C. It depends on μ, σ
D. I don't know



Solution

Answer A

Let us show that $X_{n+1}, \dots, X_{n+p}$ has the same joint distribution as $X_1, \dots, X_p$

1. Let us show that X_n has the same distribution as X_1. By induction on $n \geq 1$. True for $n = 1$
   Assume true for n i.e. X_n has the same distribution as X_1
   For any test function φ:
   $$E\big(\varphi(X_{n+1})\big)$$
   $$= pE(\varphi(X_{n+1})|U_n < p) + (1-p)E(\varphi(X_{n+1})|U_n \geq p)$$
   $$= pE\big(\varphi(X_n)\big) + (1-p)E(\varphi(X_1))$$

   because when $U_n < p$ X_{n+1} is equal to X_n and else has the same distribution as X_1. By induction hypothesis $E\big(\varphi(X_n)\big) = E\big(\varphi(X_1)\big)$ thus $E\big(\varphi(X_{n+1})\big) = E\big(\varphi(X_n)\big)$. This shows that that X_{n+1} has the same distribution as X_1.

Solution

2. The algorithm for producing $X_1, \dots, X_p$ given X_1 as input is the same as for producing $X_{n+1}, \dots, X_{n+p}$ given X_{n+1}as input; since the inputs have the same distributions, so do the outputs.

We simulate a random waypoint model with speed pdf f_{V_0}

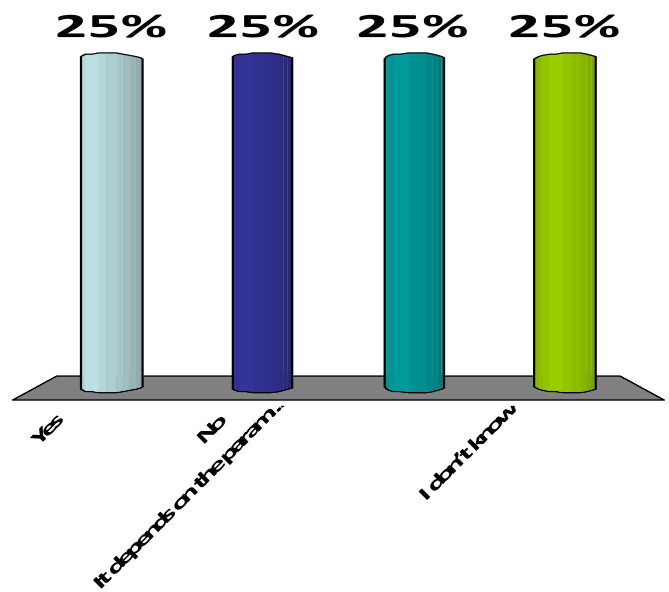
Does this simulation have a stationary regime?

A. Yes

B. No

C. It depends on the parameters of f_{V_0}

D. I don't know



Solution

There is a stationary regime if the average trip duration is finite, which depends on $\boldsymbol{f_{V_0}}$. The average trip duration is proportional to $\int_0^\infty \frac{1}{v} f_{V_0}(v)\, dv$ when the integral is finite, and is infinite otherwise.

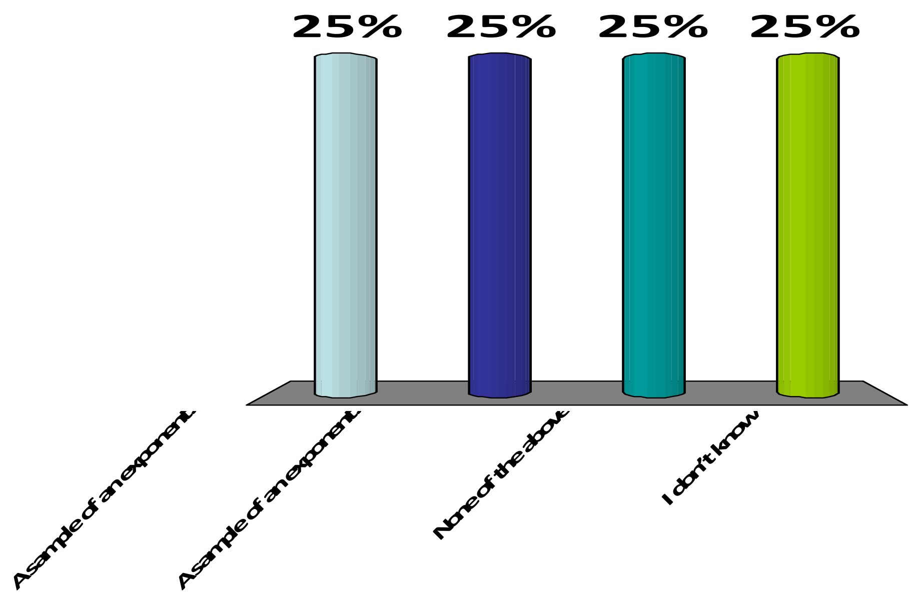
Answer C

$\text{myfun}(a) = a\ \text{randexp}(\ \ )$

where $\text{randexp}(\ \ )$ **returns one sample of the standard exponential distribution.**

What does $\text{myfun}(a)$ **return ?**

A. A sample of an exponential distribution with rate a
B. A sample of an exponential distribution with rate $1/a$
C. None of the above
D. I don't know



Solution

A sampling method for the exponential distribution with rate λ is $T = -\frac{\log U}{\lambda}$, randexp(   ) returns $-\log U$ therefore

a randexp(   ) returns T with $\lambda = 1/a$

Answer B

$a > 0$

$\text{myfun}(a) =$
do

$X =$ randn(1,1)

until $X > a$

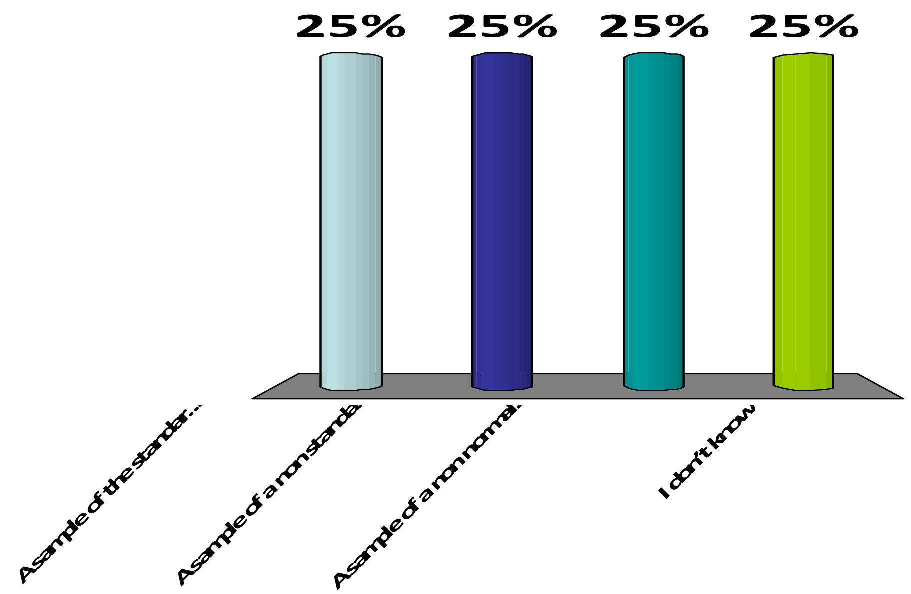
What does $\text{myfun}(a)$ **return ?**

A. A sample of the standard normal distribution $N(0,1)$

B. A sample of a non standard normal distribution $N(\mu, \sigma^2)$

C. A sample of a non normal distribution

D. I don't know



Solution

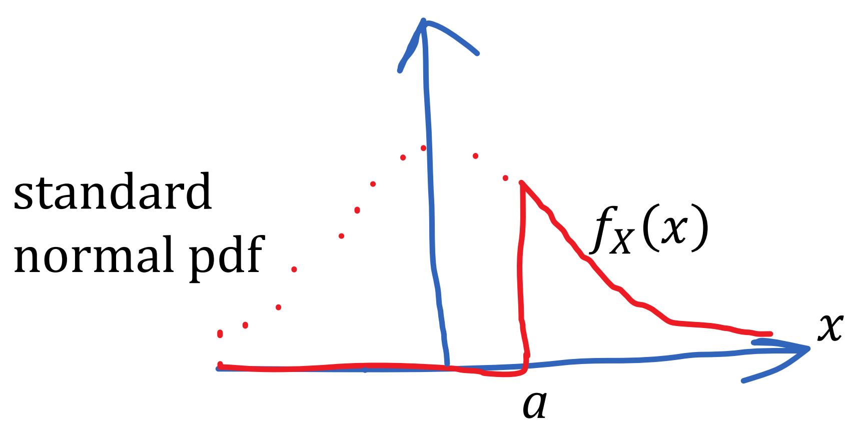
Answer C

This is rejection sampling. The output is a sample of the conditional distribution of a standard gaussian Z, given that $Z > a$

The pdf is

$f_X(x) = \eta e^{-\frac{x^2}{2}} 1_{x>a}$ where η is a normalizing constant: $\eta^{-1} = \int_a^\infty e^{-\frac{x^2}{2}} dx$

This is not a gaussian distribution



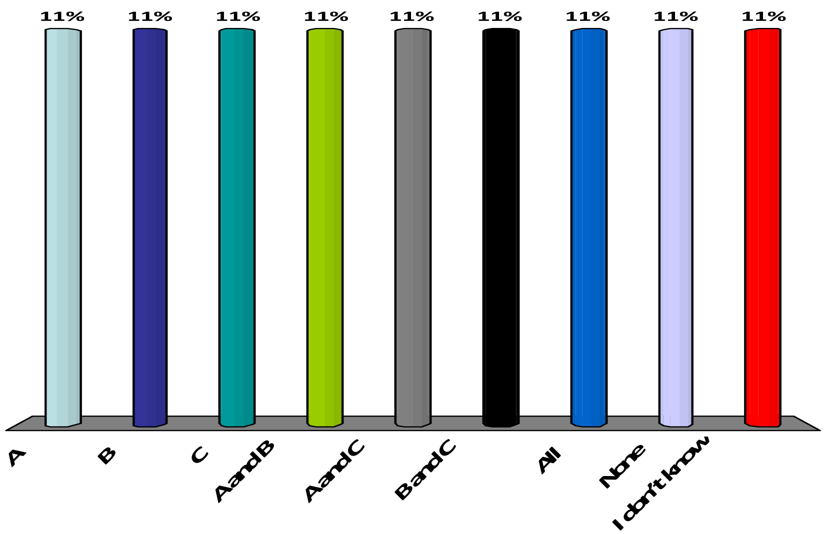
Independent outputs of a simulation are obtained by…

A. executing the runs on parallel threads using the same seed

B. executing the runs on parallel threads using truly random seeds

C. using the last RNG state of one run as seed to the next run

A. A
B. B
C. C
D. A and B
E. A and C
F. B and C
G. All
H. None
I. I don't know



Solution

B and C are correct, answer F

What does this program compute ?

(A is a subset of $[0, 1]^n$)

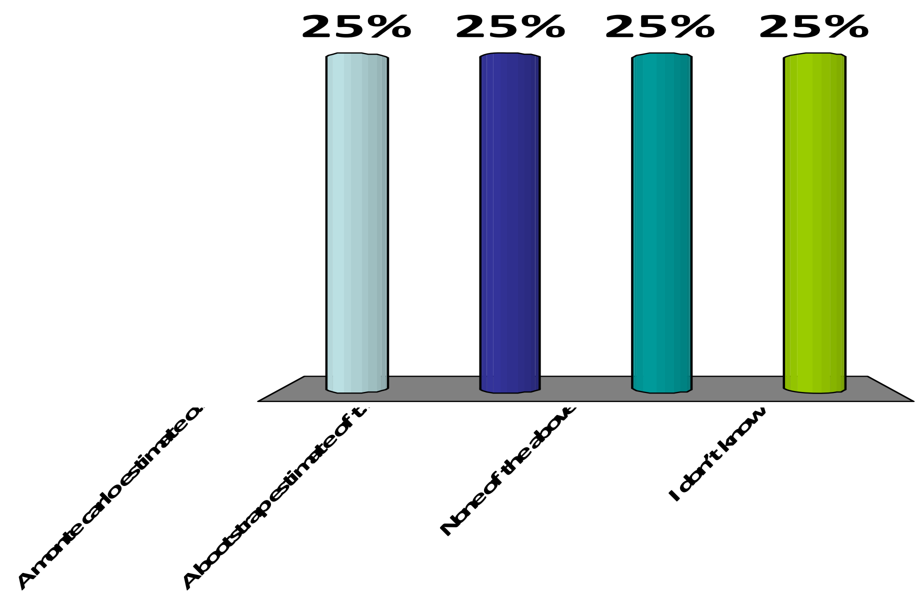
```
N=0
do r=1:R
    if (rand(n,1)∈ A)   N = N + 1
end
return(N/R)
```

A. A monte carlo estimate of vol(A)

B. A bootstrap estimate of the probability that a gaussian random vector is in A

C. None of the above

D. I don’t know



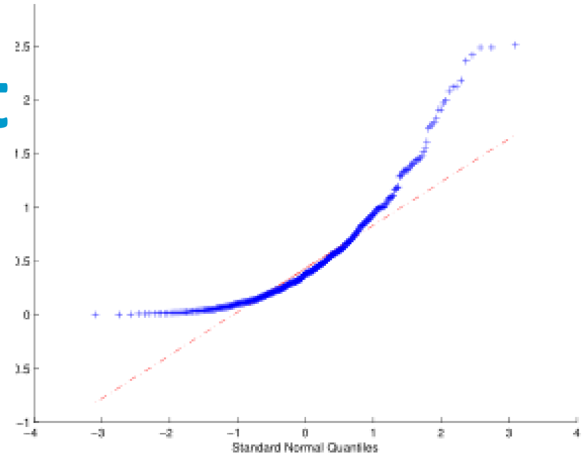
Solution

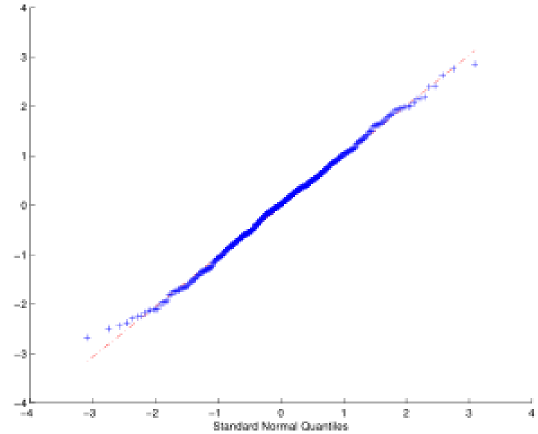
Answer A

This computes an estimate of $E(1_{X\in A}) = P(X \in A)$ where X is uniform in $[0,1]^n$

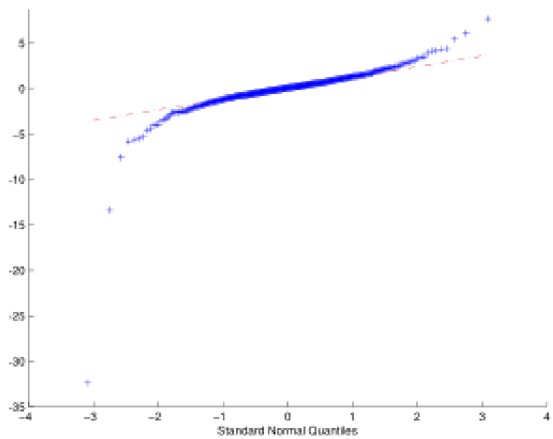
Now $P(X \in A) = \int_{[0,1]^n} 1_{x\in A} dx = \int_A dx = \text{volume}(A)$

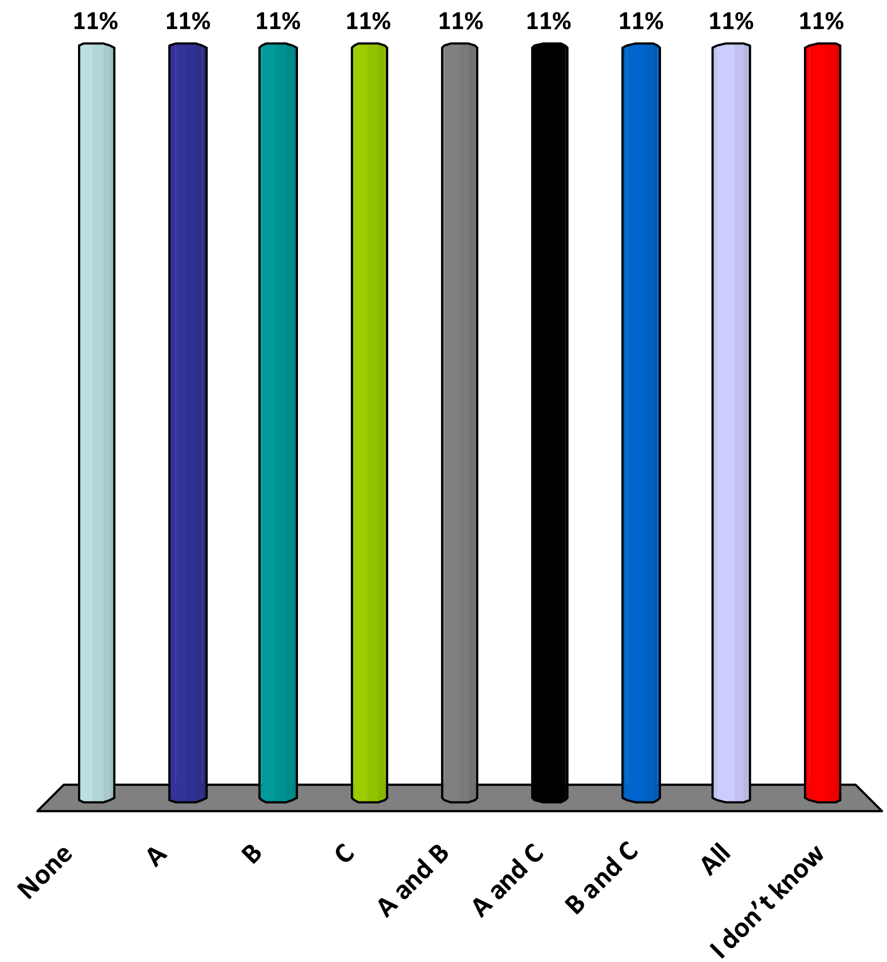
Which QQ-plot is for the exponential distribution ?





A. None
B. A
C. B
D. C
E. A and B
F. A and C
G. B and C
H. All
I. I don't know





Solution

Answer A
The positive part of the tail of the exponential distribution is heavier than for the normal distribution because the pdf decays in e^{-x} versus e^{-x^2}

The negative positive part of the tail of the exponential distribution is lighter than for the normal distribution because the exponential never has negative values

Only Plot A has these features