Forecasting Bonus Exercise

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We have a times series $Y_t$. We computed the differenced time series $X_t = Y_t - Y_{t-1}$ and found that $X_t$ can be modelled as an AR process: $X_t = \epsilon_t + 0.5 X_{t-1}$ where $\epsilon_t \sim \text{iid } N(0, \sigma^2)$

1. Is this a valid ARIMA model?
2. Compute a point forecast $\hat{X}_t(2)$
3. Compute a point forecast $\hat{Y}_t(2)$
4. Compute the first 3 terms of the impulse response of the filter $\epsilon \to Y$
5. Compute a prediction interval for $Y_{t+2}$ done at time $t$
6. How would you compute a prediction interval using the bootstrap?
1. Is this a valid ARIMA model?

A. Yes because $X = F \epsilon$ and $F$ is an ARMA filter

B. Yes because $X = F \epsilon$ and $F$ is a stable ARMA filter with stable inverse

C. No because differencing is not a stable filter

D. I don’t know
The only thing to verify is whether the filter that defines the model for $X$ is stable and has a stable inverse.

We have $X_t - 0.5 X_{t-1} = \epsilon_t$ i.e.

$$(1 - 0.5B)X = \epsilon$$

$$X = \frac{1}{1 - 0.5 B} \epsilon$$

The filter is $F = \frac{1}{1 - 0.5 B}$

The zeros of the numerator polynomial are none $\Rightarrow$ OK

The zeros of the denominator polynomial are: $z - 0.5 = 0 \Rightarrow z = 0.5, |0.5| < 1 \Rightarrow OK$

Answer B
2. The point predictions for $X$ are...

A. $\hat{X}_t(2) = X_t + 0.5\hat{X}_t(1), \quad \hat{X}_t(1) = 0.5X_t$
B. $\hat{X}_t(2) = 0.25X_t + 0.5\hat{X}_t(1), \quad \hat{X}_t(1) = 0.5X_t$
C. $\hat{X}_t(2) = 0.5X_t + 0.5\hat{X}_t(1), \quad \hat{X}_t(1) = -0.5X_t$
D. $\hat{X}_t(2) = 0.5\hat{X}_t(1), \quad \hat{X}_t(1) = 0.5X_t$
E. I don’t know
Solution

\[ X_{t+2} = \epsilon_{t+2} + 0.5 \, X_{t+1} \]
\[ X_{t+1} = \epsilon_{t+1} + 0.5 \, X_t \]

We use as point forecast the conditional expectation of \( X_{t+2} \) given we have observed \( Y \) up to time \( t \). Note that \( L \) and \( F \) are invertible, therefore observing \( Y_{1:t} \) is the same as observing \( X_{1:t} \) or \( \epsilon_{1:t} \).

Take the expectation conditional to the observation up to time \( t \) of the above equations and obtain

\[ E(X_{t+2} | Y_{1:t}) = 0.5E(X_{t+1} | Y_{1:t}) \]
\[ E(X_{t+1} | Y_{1:t}) = 0.5X_t \]

because \( E(\epsilon_{t+2} | Y_{1:t}) = E(\epsilon_{t+2} | \epsilon_{1:t}) = 0 \) and idem \( E(\epsilon_{t+1} | Y_{1:t}) = 0 \).

We can rewrite this as:

\[ \hat{X}_t(2) = 0.5\hat{X}_t(1) \]
\[ \hat{X}_t(1) = 0.5X_t \]

Answer D
3. The point predictions for $Y$ are ...

A. $\hat{Y}_t(2) = \hat{X}_t(2) + 0.5Y_t(1), \  \hat{Y}_t(1) = \hat{X}_t(1) + 0.5Y_t$

B. $\hat{Y}_t(2) = \hat{X}_t(2) - 0.5Y_t(1), \  \hat{Y}_t(1) = \hat{X}_t(1) - 0.5Y_t$

C. $\hat{Y}_t(2) = \hat{X}_t(2) + \hat{Y}_t(1), \  \hat{Y}_t(1) = \hat{X}_t(1) + Y_t$

D. $\hat{Y}_t(2) = \hat{X}_t(2) + Y_{t+1}, \  \hat{Y}_t(1) = \hat{X}_t(1) + Y_t$

E. I don’t know
Solution

We use as point forecast the conditional expectation of $Y_{t+2}$ given we have observed $Y$ (hence $X$ and $\epsilon$) up to time $t$.

$$Y_{t+2} = X_{t+2} + Y_{t+1}$$
$$Y_{t+1} = X_{t+1} + Y_{t}$$

Take the expectation conditional to the observation up to time $t$ and obtain

$$\hat{Y}_t(2) = \hat{X}_t(2) + \hat{Y}_t(1)$$
$$\hat{Y}_t(1) = \hat{X}_t(1) + Y_t$$

Answer C
4. What is the impulse response of the filter $\varepsilon \to Y$?

A. 1.000 -1.000 -1.500 ...
B. 1.000 -1.500 -2.250 ...
C. 1.000 1.500 1.750 ...
D. None of the above
E. I don’t know
Solution

Answer C
We have $Y_t - Y_{t-1} = X_t$, i.e. $(1 - B)Y = X$
Further, $X = \frac{1}{1 - 0.5B} \epsilon$

Therefore $Y = \frac{1}{(1 - B)(1 - 0.5B)}$

The impulse response can be computed by power series calculus

$$\frac{1}{(1 - B)(1 - 0.5B)} = (1 + B + B^2 + \cdots)(1 + 0.5B + 0.25B^2 + \cdots)$$

$$= (1 + 1.5B + 1.75B^2 + \cdots)$$

or with matlab

```matlab
>> h=filter([1],[1 -1],filter([1],[1 -0.5],[1 0 0 0 0 0 0]))
```

```
h =
   1.0000   1.5000   1.7500   1.8750   1.9375   1.9688   1.9844
```
5. A prediction interval for $Y_{t+2}$ done at $t$ is ...

A. $\hat{Y}_t(2) \pm 1.96 \times \sigma$
B. $\hat{Y}_t(2) \pm 1.96 \times \sqrt{1.25}\sigma$
C. $\hat{Y}_t(2) \pm 1.96 \times \sqrt{2.25}\sigma$
D. $\hat{Y}_t(2) \pm 1.96 \times \sqrt{3.25}\sigma$
E. $\hat{Y}_t(2) \pm 1.96 \times \sqrt{4.25}\sigma$
F. $\hat{Y}_t(2) \pm 1.96 \times \sqrt{5.25}\sigma$
G. I don’t know
Solution

Answer D.

We have $Y_{t+2} = \epsilon_{t+2} + 1.5 \epsilon_{t+1} + 1.75\epsilon_t + 1.875\epsilon_{t-1} + \ldots$ \hspace{1em} (eq. 1)

This is not a good formula for computing $Y_{t+2}$ out of the complete series $\epsilon_t$ because the coefficients become large (the filter $\frac{1}{1-B}$ is unstable) and the error accumulates. It is better to use

\[
\begin{align*}
Y_{t+2} &= X_{t+2} + Y_{t+1} \\
X_{t+2} &= \epsilon_{t+2} + 0.5X_{t+1} \\
Y_{t+1} &= X_{t+1} + Y_t \\
X_{t+1} &= \epsilon_{t+1} + 0.5X_t
\end{align*}
\]

as we did earlier in order to compute the point forecasts.
\[ Y_{t+2} = \epsilon_{t+2} + 1.5 \epsilon_{t+1} + 1.75\epsilon_t + 1.875\epsilon_{t-1} + \cdots \quad (eq. 1) \]

However, (eq. 1) can be used to simplify the computation of prediction intervals. Observe that the red box is necessarily \( \hat{Y}_t(2) \) – to see why, take the conditional expectation given \( Y_{1:t} \).

In other words (Innovation Formula):
\[ Y_{t+2} = \epsilon_{t+2} + 1.5 \epsilon_{t+1} + \hat{Y}_t(2) \quad (eq. 2) \]
which can be used to produce prediction intervals. Conditional to the observation up to time \( t \), \( \hat{Y}_t(2) \) is known (non random) and \( \epsilon_{t+2}, \epsilon_{t+1} \) are iid \( N(0, \sigma^2) \), hence \( \epsilon_{t+2} + 1.5 \epsilon_{t+1} \) is \( N(0, \nu) \) with
\[ \nu = \sigma^2 + (1.5)^2 \sigma^2 = 3.25 \sigma^2 \]

Therefore a 95%-prediction interval for \( Y_{t+2} \) done at time \( t \) is
\[ \hat{Y}_t(2) \pm 1.96 \times \sqrt{3.25\sigma} \]
6. Which is a correct implementation of the bootstrap for computing 95%-prediction intervals at time $t$ and lag 2?

A. A
Compute the time series $\epsilon_s = X_s - 0.5X_{s-1}$, $s = 3: t$
do $r = 1: 999$
\begin{align*}
draw & e_r^r, s = 3: (t + 2) \text{ with replacements from } \epsilon_s, s = 3: t \\
\text{compute } & X_{1:t}^r, Y_{1:t}^r \text{ and } \hat{Y}_t^r(2) \text{ using } X_s^r = e_s^r + 0.5X_{s-1}^r, \\
& Y_s^r = X_s^r + Y_{s-1}^r \text{ and the formula for } \hat{Y}_t^r(2) \\
Y_{t+2}^r = & e_{t+2}^r + 1.5e_{t+1}^r + \hat{Y}_t^r(2) \\
\end{align*}
Prediction interval is $[Y^{(25)}_{t+\ell}, Y^{(975)}_{t+\ell}]$

B. B
Compute the time series $\epsilon_s = X_s - 0.5X_{s-1}$, $s = 3: t$
do $r = 1: 999$
\begin{align*}
draw & e_1^r, e_2^r \text{ with replacements from } \epsilon_s, s = 3: t \\
Y_{t+2}^r = & e_1^r + 1.5e_2^r + \hat{Y}_t(2) \\
\end{align*}
Prediction interval is $[Y^{(25)}_{t+\ell}, Y^{(975)}_{t+\ell}]$
Solution

A is simulating the entire time series, therefore it is producing a sample of the unconditional distribution of $Y_{t+2}$. It is not the prediction, it is what can be said about $Y_{t+2}$ for an observer who knows the statistics of the time series but did not observe $Y_1, \ldots, Y_t$.

B is simulating the time series from $t + 1$ to $t + 2$ given the data up to time $t$, therefore it is producing a sample of the conditional distribution of $Y_{t+2}$ given the observed past. It is a correct implementation.

Answer B