Forecasting Bonus Exercise

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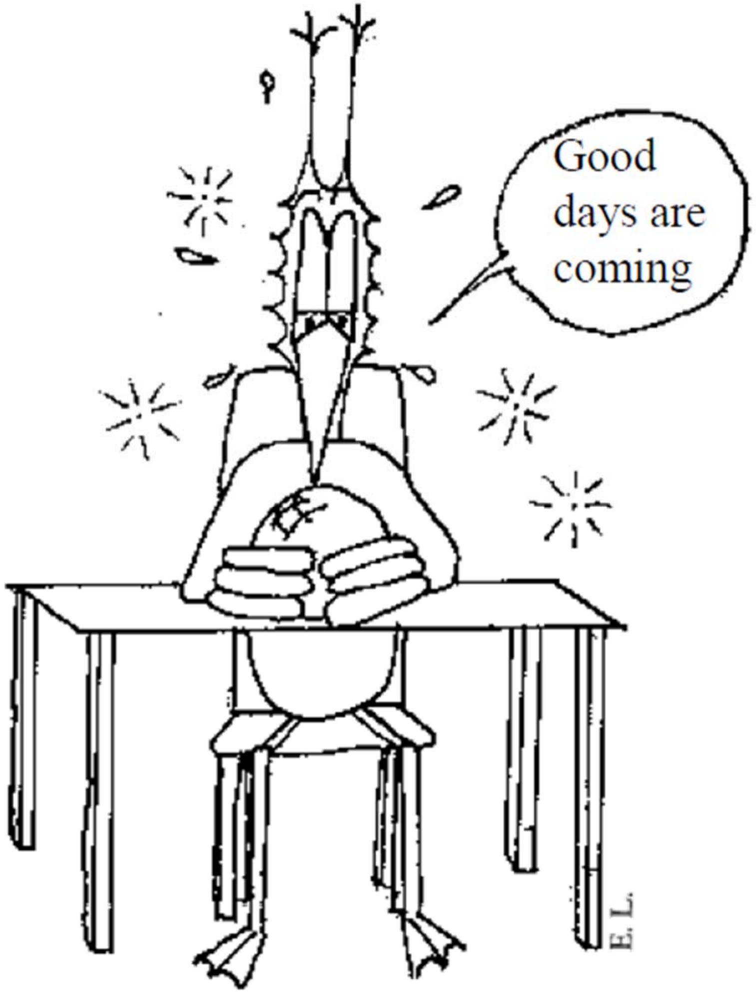

Exercise: Compute the prediction

We have a times series Y_t . We computed the differenced time series $X_t = Y_t - Y_{t-1}$ and found that X_t can be modelled as an AR process: $X_t = \epsilon_t + 0.5\ X_{t-1}$ where $\epsilon_t \sim$ iid $N(0, \sigma^2)$

1. Is this a valid ARIMA model ?
2. Compute a point forecast $\hat{X}_t(2)$
3. Compute a point forecast $\hat{Y}_t(2)$
4. Compute the first 3 terms of the impulse response of the filter $\epsilon \rightarrow Y$
5. Compute a prediction interval for Y_{t+2} done at time t
6. How would you compute a prediction interval using the boostrap ?

1. Is this a valid ARIMA model ?

A. Yes because X=$F\epsilon$ and F is an ARMA filter

B. Yes because $X = F\epsilon$ and F is a stable ARMA filter with stable inverse

C. No because differencing is not a stable filter

D. I don't know

Exercise: Compute the prediction

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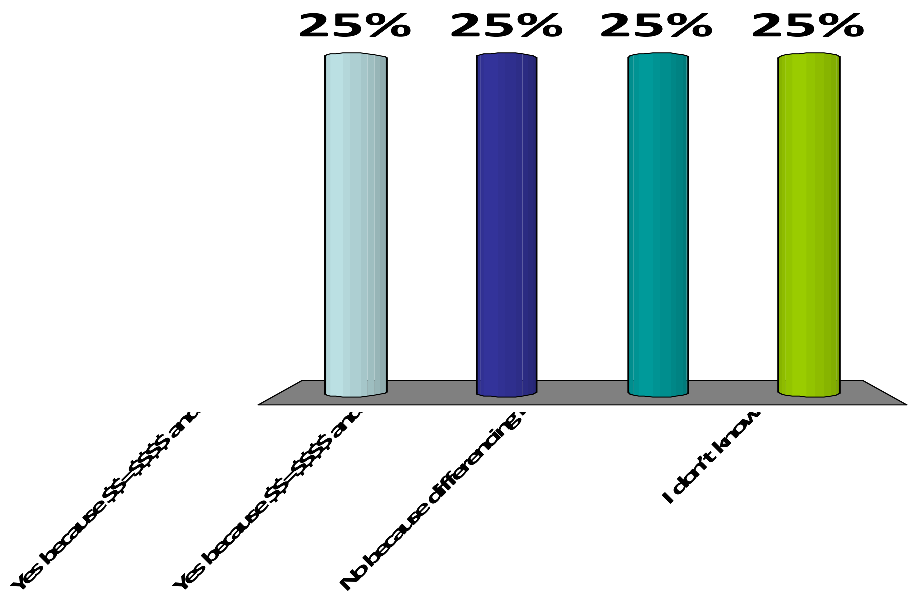
1. Is this a valid ARIMA model ?
2. Compute the prediction formulae



Solution

The only thing to verify is whether the filter that defines the model for X is stable and has a stable inverse.

We have $X_t - 0.5\, X_{t-1} \;= \epsilon_t$ i.e.

$$(1 - 0.5B)X = \epsilon$$

$$X = \frac{1}{1 - 0.5\,B}\epsilon$$

The filter is $F = \frac{1}{1-0.5\,B}$

The zeros of the numerator polynomial are none $\Rightarrow$ OK

The zeros of the denominator polynomial are : $z - 0.5 \;= 0 \Rightarrow z = 0.5, |0.5| < 1 \Rightarrow$ OK

Answer B

2. The point predictions for X are…

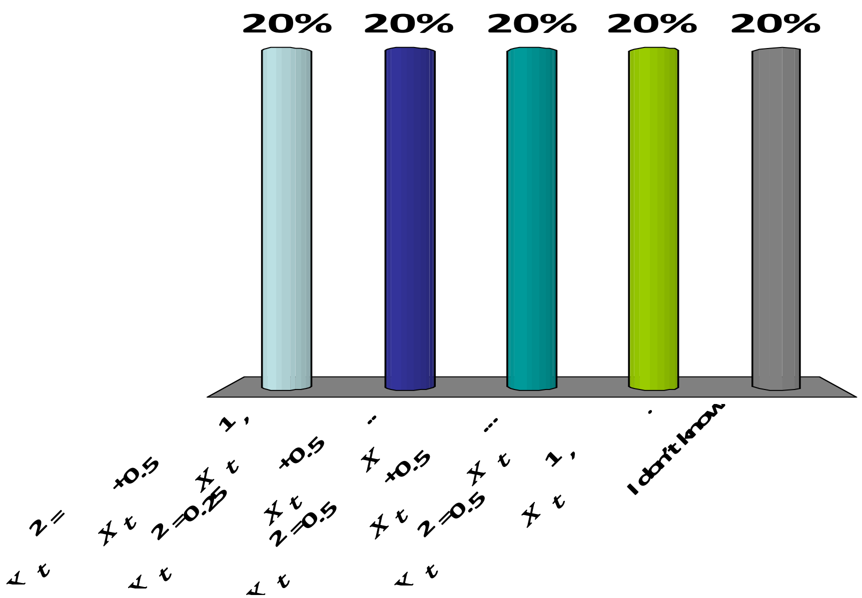
A. $\hat{X}_t(2) = X_t + 0.5\hat{X}_t(1),\ \ \hat{X}_t(1) = 0.5X_t$

B. $\hat{X}_t(2) = 0.25X_t + 0.5\hat{X}_t(1),\ \ \hat{X}_t(1) = 0.5X_t$

C. $\hat{X}_t(2) = 0.5X_t + 0.5\hat{X}_t(1),\ \ \hat{X}_t(1) = -0.5X_t$

D. $\hat{X}_t(2) = 0.5\hat{X}_t(1),\ \ \hat{X}_t(1) = 0.5X_t$

E. I don’t know



Solution

$$X_{t+2} = \epsilon_{t+2} + 0.5\, X_{t+1}$$
$$X_{t+1} = \epsilon_{t+1} + 0.5\, X_t$$

We use as point forecast the conditional expectation of X_{t+2} given we have observed Y up to time t. Note that L and F are invertible, therefore observing $Y_{1:t}$ is the same as observing $X_{1:t}$ or $\epsilon_{1:t}$.
Take the expectation conditional to the observation up to time t of the above equations and obtain

$$E(X_{t+2}|Y_{1:t}) = 0.5E(X_{t+1}|Y_{1:t})$$
$$E(X_{t+1}|Y_{1:t}) = 0.5X_t$$

because $E(\epsilon_{t+2}|Y_{1:t}) = E(\epsilon_{t+2}|\epsilon_{1:t}) = 0$ and idem $E(\epsilon_{t+1}|Y_{1:t}) = 0$.
We can rewrite this as:

$$\hat{X}_t(2) = 0.5\hat{X}_t(1)$$
$$\hat{X}_t(1) = 0.5X_t$$

Answer D

3. The point predictions for Y are …

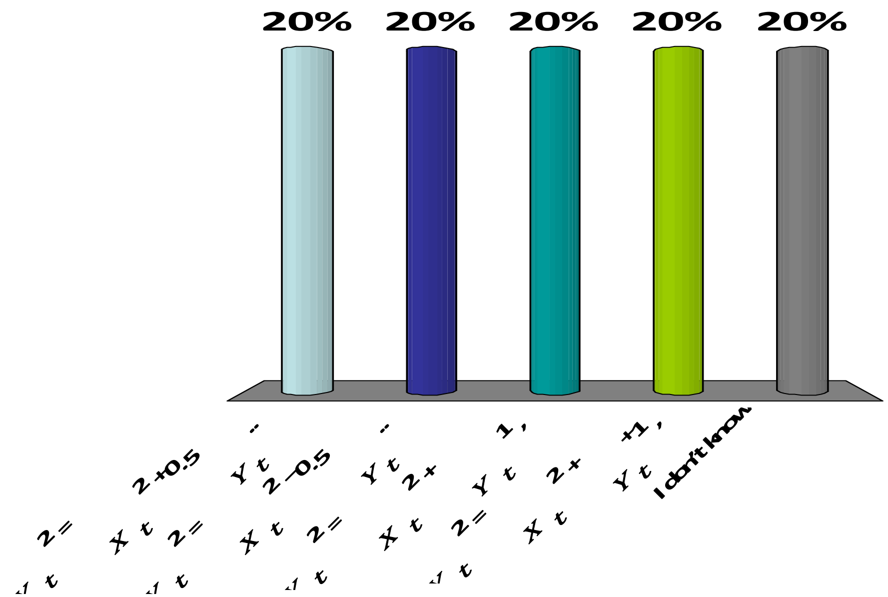
A. $\hat{Y}_t(2) = \hat{X}_t(2) + 0.5Y_t(1),\ \hat{Y}_t(1) = \hat{X}_t(1) + 0.5Y_t$

B. $\hat{Y}_t(2) = \hat{X}_t(2) - 0.5Y_t(1),\ \hat{Y}_t(1) = \hat{X}_t(1) - 0.5Y_t$

C. $\hat{Y}_t(2) = \hat{X}_t(2) + \hat{Y}_t(1),\ \hat{Y}_t(1) = \hat{X}_t(1) + Y_t$

D. $\hat{Y}_t(2) = \hat{X}_t(2) + Y_{t+1},\ \hat{Y}_t(1) = \hat{X}_t(1) + Y_t$

E. I don't know



Solution

We use as point forecast the conditional expectation of Y_{t+2} given we have observed Y (hence X and ϵ) up to time t.

$$Y_{t+2} = X_{t+2} + Y_{t+1}$$
$$Y_{t+1} = X_{t+1} + Y_t$$

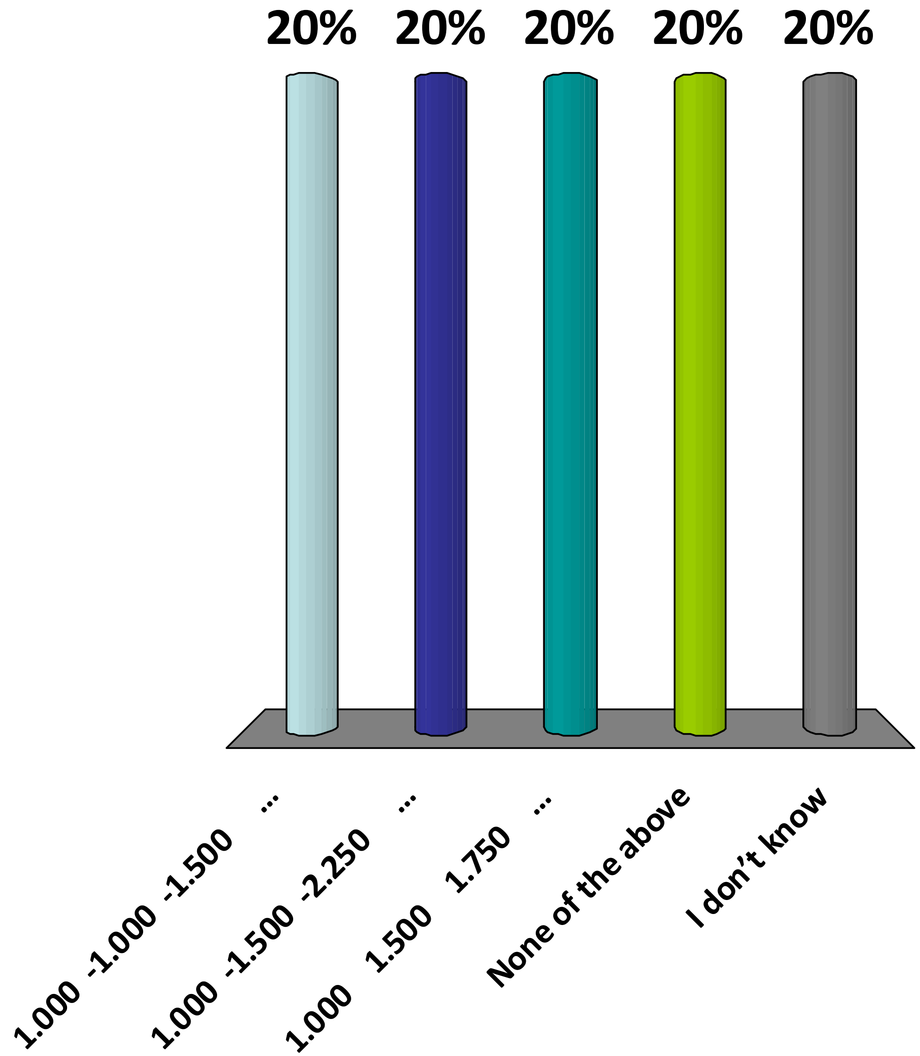
Take the expectation conditional to the observation up to time t and obtain

$$\hat{Y}_t(2) = \hat{X}_t(2) + \hat{Y}_t(1)$$
$$\hat{Y}_t(1) = \hat{X}_t(1) + Y_t$$

Answer C

4. What is the impulse response of the filter $\epsilon \rightarrow Y$?

A. 1.000 -1.000 -1.500 ...
B. 1.000 -1.500 -2.250 ...
C. 1.000 1.500 1.750 ...
D. None of the above
E. I don’t know



Solution

Answer C

We have $Y_t - Y_{t-1} = X_t$, i.e. $(1-B)Y = X$

Further, $X = \frac{1}{1-0.5B}\epsilon$

Therefore $Y = \frac{1}{(1-B)(1-0.5B)}$

The impulse response can be computed by power series calculus

$$\frac{1}{(1-B)(1-0.5B)} = (1 + B + B^2 + \cdots)(1 + 0.5B + 0.25B^2 + \cdots)$$
$$= (1 + 1.5B + 1.75B^2 + \cdots)$$

or with matlab

```
>> h=filter([1],[1 -1],filter([1],[1 -0.5],[1 0 0 0  0 0 0]))
h =
    1.0000    1.5000    1.7500    1.8750    1.9375    1.9688    1.9844
```

5. A prediction interval for Y_{t+2} done at t is …

A. $\hat{Y}_t(2) \pm 1.96 \times \sigma$

B. $\hat{Y}_t(2) \pm 1.96 \times \sqrt{1.25}\sigma$

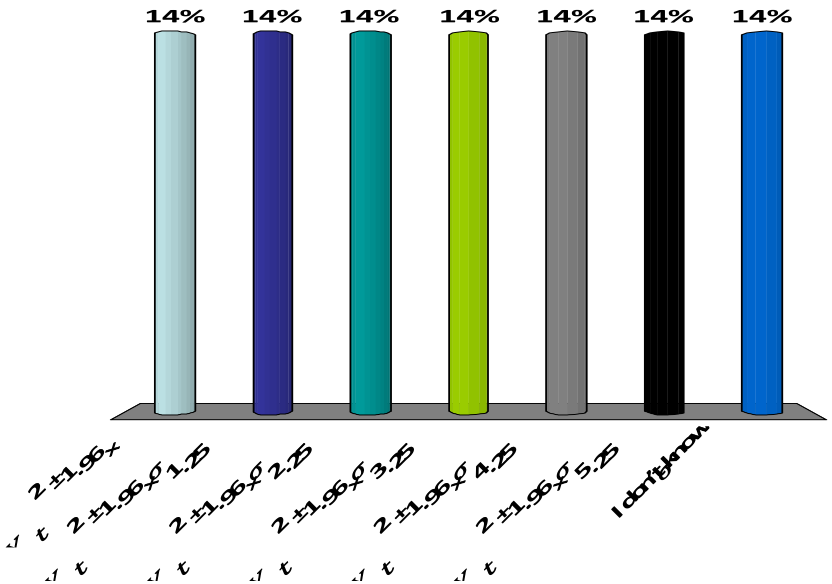
C. $\hat{Y}_t(2) \pm 1.96 \times \sqrt{2.25}\sigma$

D. $\hat{Y}_t(2) \pm 1.96 \times \sqrt{3.25}\sigma$

E. $\hat{Y}_t(2) \pm 1.96 \times \sqrt{4.25}\sigma$

F. $\hat{Y}_t(2) \pm 1.96 \times \sqrt{5.25}\sigma$

G. I don't know



Solution

Answer D.

We have $Y_{t+2} = \epsilon_{t+2} + 1.5\ \epsilon_{t+1} + 1.75\epsilon_t + 1.875\epsilon_{t-1} + \cdots \quad (eq. 1)$

This is not a good formula for computing Y_{t+2} out of the complete series ϵ_t because the coefficients become large (the filter $\frac{1}{1-B}$ is unstable) and the error accumulates. It is better to use

$$Y_{t+2} = X_{t+2} + Y_{t+1}$$
$$X_{t+2} = \epsilon_{t+2} + 0.5X_{t+1}$$
$$Y_{t+1} = X_{t+1} + Y_t$$
$$X_{t+1} = \epsilon_{t+1} + 0.5X_t$$

as we did earlier in order to compute the point forecasts.

Solution

$$Y_{t+2} = \epsilon_{t+2} + 1.5\,\epsilon_{t+1} + \boxed{1.75\epsilon_t + 1.875\epsilon_{t-1} + \cdots} \quad (eq.1)$$

However, (eq. 1) can be used to simplify the computation of prediction intervals. Observe that the red box is necessarily $\hat{Y}_t(2)$ – to see why, take the conditional expectation given $Y_{1:t}$.

In other words (Innovation Formula):

$$Y_{t+2} = \epsilon_{t+2} + 1.5\,\epsilon_{t+1} + \hat{Y}_t(2) \quad (eq\,.2)$$

which can be used to produce prediction intervals. Conditional to the observation up to time t, $\hat{Y}_t(2)$ is known (non random) and $\epsilon_{t+2}, \epsilon_{t+1}$ are iid $N(0, \sigma^2)$, hence $\epsilon_{t+2} + 1.5\,\epsilon_{t+1}$ is $N(0, v)$ with

$$v = \sigma^2 + (1.5)^2\sigma^2 = 3.25\,\sigma^2$$

Therefore a 95%-prediction interval for Y_{t+2} done at time t is

$$\hat{Y}_t(2) \pm 1.96 \times \sqrt{3.25}\sigma$$

6. Which is a correct implementation of the bootstrap for computing 95%-prediction intervals at time t and lag 2 ?

A. A

B. B

C. Both

D. None

E. I don't know

A

Compute the time series $\epsilon_s = X_s - 0.5X_{s-1}$, $s = 3{:}\,t$

do $r = 1{:}\,999$ {

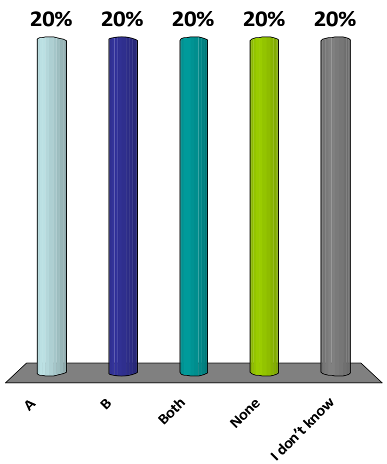
draw $e_s^r, s = 3{:}\,(t+2)$ with replacements from $\epsilon_s, s = 3{:}\,t$

compute $X_{1:t}^r, Y_{1:t}^r$ and $\hat{Y}_t^r(2)$ using $X_s^r = e_s^r + 0.5X_{s-1}^r$,

$Y_s^r = X_s^r + Y_{s-1}^r$ and the formula for $\hat{Y}_t^r(2)$

$Y_{t+2}^r = e_{t+2}^r + 1.5e_{t+1}^r + \hat{Y}_t^r(2)$}

Prediction interval is $[Y_{t+\ell}^{(25)}, Y_{t+\ell}^{(975)}]$



B

Compute the time series $\epsilon_s = X_s - 0.5X_{s-1}$, $s = 3{:}\,t$

do $r = 1{:}\,999$ {

draw e_1^r, e_2^r with replacements from $\epsilon_s, s = 3{:}\,t$

$Y_{t+2}^r = e_1^r + 1.5e_2^r + \hat{Y}_t(2)$}

Prediction interval is $[Y_{t+\ell}^{(25)}, Y_{t+\ell}^{(975)}]$

Solution

A is simulating the entire time series, therefore it is producing a sample of the unconditional distribution of Y_{t+2}. It is not the prediction, it is what can be said about Y_{t+2} for an observer who knows the statistics of the time series but did not observe $Y_1, \ldots, Y_t$

B is simulating the time series from $t + 1$ to $t + 2$ given the data up to time t, therefore it is producing a sample of the conditional distribution of Y_{t+2} given the observed past. It is a correct implementation.

Answer B