Palm Calculus
Part 2
Theory

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1. Palm Calculus: Framework

- A stationary process (simulation) with state $S_t$.
- Some quantity $X_t$ measured at time $t$. Assume that $(S_t;X_t)$ is jointly stationary.

I.e., $S_t$ is in a stationary regime and $X_t$ depends on the past, present and future state of the simulation in a way that is invariant by shift of time origin.

Examples

- $S_t = \text{current position of mobile, speed, and next waypoint}$
- *Jointly stationary with $S_t$: $X_t = \text{current speed at time } t$; $X_t = \text{time to be run until next waypoint}$
- *Not jointly stationary with $S_t$: $X_t = \text{time at which last waypoint occurred}$
Consider some **selected transitions** of the simulation, occurring at times $T_n$.

- Example: $T_n =$ time of $n^{th}$ trip end

Formally, a stationary point process in our setting is associated with a subset $\mathcal{F}_0$ of the set of all possible state transitions of the simulation. It is made of all time instants $t$ at which the simulator does a transition in $\mathcal{F}_0$, i.e. such that $(S_{t^-}, S_{t^+}) \in \mathcal{F}_0$.

- $T_n$ is a called a **stationary point process** associated to $S_t$
  - Stationary because $S_t$ is stationary
  - Jointly stationary with $S_t$

- The time instants of the point process $T_n$ are such that $... < T_{-2} < T_{-1} < T_0 \leq 0 < T_1 < T_2 < ...$

- Time 0 is the *arbitrary* point in time
Palm Expectation

- Assume: $X_t, S_t$ are jointly stationary, $T_n$ is a stationary point process associated with $S_t$

- **Definition**: the Palm Expectation is

$$E^t(X_t) = E(X_t | \text{a selected transition occurred at time } t)$$

- By stationarity:

$$E^t(X_t) = E^0(X_0)$$

- Example:
  - $T_n = \text{time of n}\text{th trip end}, \ X_t = \text{instant speed at time } t$
  - $E^t(X_t) = E^0(X_0) = \text{average speed observed at a waypoint}$
\[ E(X_t) = E(X_0) \] expresses the **time average** viewpoint.

\[ E^t(X_t) = E^0(X_0) \] expresses the **event average** viewpoint.

Example for random waypoint:

- \( T_n = \) time of \( n^{th} \) trip end, \( X_t = \) instant speed at time \( t \)
- \( E^t(X_t) = E^0(X_0) = \) average speed observed at trip end
- \( E(X_t) = E(X_0) = \) average speed observed at an arbitrary point in time
**Formal Definition**

- **In discrete time**, we have an elementary conditional probability

\[ E^t(Y) = \mathbb{E}(Y | N(t) = 1) = \frac{\mathbb{E}(YN(t))}{\mathbb{E}(N(t))} = \frac{\mathbb{E}(YN(t))}{\mathbb{P}(N(t) = 1)} \]

- **In continuous time**, the definition is a little more sophisticated
  - uses Radon Nikodym derivative—see lecture note for details
  - Also see [BaccelliBremaud87] for a formal treatment

- **Palm probability** is defined similarly

The Palm *probability* is defined similarly, namely

\[ \mathbb{P}^0(X(0) \in W) = \mathbb{P}(X(0) \in W | \text{ a point occurs at time } 0) \]

Note that \( \mathbb{P}^0(T_0 = 0) = 1 \), i.e., under the Palm probability, \( T_0 \) is 0 with probability 1.
Ergodic Interpretation

Assume simulation is stationary + ergodic, i.e. sample path averages converge to expectations; then we can estimate time and event averages by:

\[ \mathbb{E}(X_0) = \lim_{T \to +\infty} \frac{1}{T} \sum_{s=1}^{T} X_s \]

\[ \mathbb{E}^0(X_0) = \lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} X_{T_n} \]

In terms of probabilities:

- Stationary probability:
  \[ \mathbb{P}(X_t \in W) \approx \text{fraction of time that } X_t \text{ is in some set } W \]

- Palm probability:
  \[ \mathbb{P}^t(X_t \in W) \approx \text{fraction of selected transitions at which } X_t \text{ is in } W \]
Intensity of a Stationary Point Process

Intensity of selected transitions: $\lambda := \text{expected number of transitions per time unit}$

**Intensity** The intensity $\lambda$ of the point process is defined as the expected number of points per time unit. We have assumed that there cannot be two points at the same instant. In discrete or continuous time, the intensity $\lambda$ is defined as the unique number such that the number $N(t, t + \tau)$ of points during any interval $[t, t + \tau]$ satisfies [4]:

$$\mathbb{E}(N(t, t + \tau)) = \lambda \tau \quad (7.16)$$

In discrete time, $\lambda$ is also simply equal to the probability that there is a point at an arbitrary time:

$$\lambda = \mathbb{P}(T_0 = 0) = \mathbb{P}(N(0) = 1) = \mathbb{P}(N(t) = 1) \quad (7.17)$$

where the latter is valid for any $t$, by stationarity.

One can think of $\lambda$ as the (average) rate of the event clock.
Two Palm Calculus Formulae

- **Intensity Formula:**

\[
\frac{1}{\lambda} = \mathbb{E}^0(T_1 - T_0) = \mathbb{E}^0(T_1)
\]

where by convention $T_0 \leq 0 < T_1$

- **Inversion Formula**

\[
\mathbb{E}(X_t) = \mathbb{E}(X_0) = \lambda \mathbb{E}^0 \left( \int_0^{T_1} X_s \, ds \right)
\]

- The proofs are simple in discrete time – see lecture notes
Example 7.6: Gatekeeper, continued. Assume we model the gatekeeper example as a discrete event simulation, and consider as point process the waking ups of the gatekeeper. Let $X(t)$ be the execution time of a hypothetical job that would arrive at time $t$. The average job execution time, sampled with the standard clock (customer viewpoint) is

$$W_c = \mathbb{E}(X(t)) = \mathbb{E}(X(0))$$

whereas the average execution time, sampled with the event clock (system designer viewpoint), is

$$W_s = \mathbb{E}^t(X(t)) = \mathbb{E}^0(X(0))$$

The inversion formula gives

$$W_c = \lambda \mathbb{E}^0 \left( \int_0^{T_1} X(t) dt \right) = \lambda \mathbb{E}^0(X(0)T_1)$$

(recall that $T_0 = 0$ under the Palm probability and $X(0)$ is the execution time for a job that arrives just after time 0). Let $C$ be the cross-covariance between sleep time and execution time:

$$C := \mathbb{E}^0(T_1X(0)) - \mathbb{E}^0(T_1)\mathbb{E}^0(X(0))$$

then

$$W_c = \lambda [C + \mathbb{E}^0(X(0))\mathbb{E}^0(T_1)]$$

By the inversion formula $\lambda = \frac{1}{\mathbb{E}^0(T_1)}$ thus

$$W_c = W_s + \lambda C$$

which is the formula we had derived using the heuristic in Section 7.1.
The interval between 2 buses is ~ $U(15, 25)$ minutes

A. There are 2 buses in average per hour
B. There are 3 buses in average per hour
C. There are 4 buses in average per hour
D. None of the above
E. I don’t know
Solution

Intensity formula \( \Rightarrow \lambda = \frac{1}{E^0(T_1-T_0)} = \frac{1}{20} mn^{-1} = 3 h^{-1} \)

Answer B
The validity of the formula in the previous question requires that ...

A. The arrival process is Poisson
B. The arrival process is stationary
C. The interarrival times are iid
D. None of the above
E. I don’t know

![Bar chart showing responses]

- 57% chose A.
- 29% chose C.
- 7% chose B.
- 7% chose D.
- 0% chose E.
Solution

Answer B. Palm calculus requires only stationarity. Note that a Poisson process is stationary, as is a process with independent interarrival times.
2. Other Palm Calculus Formulae

**Theorem 7.3.** Let $X(t) = T^+(t) - t$ (time until next point, also called residual time), $Y(t) = t - T^-(t)$ (time since last point), $Z(t) = T^+(t) - T^-(t)$ (duration of current interval). For any $t$, the distributions of $X(t)$ and $Y(t)$ are equal, with PDF:

$$f_X(s) = f_Y(s) = \lambda \mathbb{I}_0^+(T_1 > s) = \lambda \int_s^{+\infty} f_T^0(u) du$$

(7.28)

where $f_T^0$ is the Palm PDF of $T_1 - T_0$ (PDF of inter-arrival times). The PDF of $Z(t)$ is

$$f_Z(s) = \lambda s f_T^0(s)$$

(7.29)

In particular, it follows that

$$\mathbb{E}(X(t)) = \mathbb{E}(Y(t)) = \frac{\lambda}{2} \mathbb{E}^0(T_1^2) \quad \text{in continuous time}$$

(7.30)

$$\mathbb{E}(X(t)) = \mathbb{E}(Y(t)) = \frac{\lambda}{2} \mathbb{E}^0(T_1(T_1 + 1)) \quad \text{in discrete time}$$

(7.31)

$$\mathbb{E}(Z(t)) = \lambda \mathbb{E}^0(T_1^2)$$

(7.32)
Joe’s Waiting Time

\[ E(X(t)) = \mathbb{E}(Y_t) = \frac{\lambda}{2} \mathbb{E}^0(T_1^2) \]

- \( E(X(t)) \) = system’s viewpoint
- \( \mathbb{E}(Y_t) \) = penalty due to variability

\[ E(X(t)) = \frac{\lambda}{2} E^0(T_1^2) = \frac{\lambda}{2} \left( E^0(T_1) \right)^2 + \frac{\lambda}{2} \text{var}^0(T_1) \]

mean waiting time = \( \frac{1}{2} E^0(T_1) + \frac{\lambda}{2} \text{var}^0(T_1) \)

0.5 \times \text{mean time between buses} \\
\text{system’s viewpoint}
Feller’s Paradox

At bus stop in average $\lambda$ buses per hour. Inspector measures time between all bus inter-departures. Inspector estimates $\mathbb{E}^0(T_1 - T_0) = \frac{1}{\lambda}$

Joe arrives at time $t$ and measures $X_t = (\text{time until next bus} - \text{time since last bus})$. Joe estimates $\mathbb{E}(X_0) = \mathbb{E}(T_1 - T_0)$

Inversion formula:

$$\mathbb{E}(T_1 - T_0) = \lambda \mathbb{E}^0\left(\int_0^{T_1} X_t dt\right) = \lambda \mathbb{E}^0(T_1^2) = \frac{1}{\lambda} + \lambda \text{var}^0(T_1 - T_0)$$

Joe’s estimate always larger than Inspector’s (Feller’s Paradox)
We encountered Feller’s Paradox Already
For a Poisson process, what is the distribution of the length of an interval?

**Example 11.6: Poisson Process.** Assume that $T_n$ is a Poisson process (a very special case). We have $f_T(t) = \lambda e^{-\lambda s}$ and $\mathbb{P}^0(T_1 > s) = \mathbb{P}^0(T_1 \geq s) = e^{-\lambda s}$ thus $f_X(s) = f_Y(s) = f_T(s)$, as expected.

The distribution of $Z_t$ has density

$$f_T(s) = \lambda^2 s e^{-\lambda s}$$

i.e., it is an Erlang-2 distribution.
THEOREM 7.6 (Little’s Formula). The mean number of customers in the system at time $t$, $\bar{N} := \mathbb{E}(N(t))$, is independent of $t$ and satisfies

$$\bar{N} = \lambda \bar{R}$$

where $\lambda$ is the arrival rate and $\bar{R}$ the average response time, experienced by an arbitrary customer.
A sensor senses events; the sensing interval is $\sim N(\mu, \sigma^2)$. An engineer comes and checks the current sensing interval. In average, she finds…

A. $\mu + \sigma^2$
B. $\mu(1 + \frac{\sigma^2}{\mu^2})$
C. $\mu(1 + \sigma^2)$
D. $\frac{1}{\mu} (1 + \frac{\sigma^2}{\mu})$
E. $\frac{1}{\mu} (1 + \frac{\sigma^2}{\mu^2})$
F. I don’t know
Solution

By the Feller paradox formula (7-32), she finds

\[ E(Z(t)) = \lambda E^0(T_1^2) \]

\[ = \frac{1}{\mu} \times (\mu^2 + \sigma^2) \]

\[ = \mu \left(1 + \frac{\sigma^2}{\mu^2}\right) \]

Answer B

It is also the only formula where the units are OK
A sensor senses events; the sensing interval is $\sim \text{expo}(\lambda)$, i.e. the event process is Poisson. An engineer comes and checks the current sensing interval. In average, she finds...

A. $\frac{1}{\lambda}$

B. $\frac{2}{\lambda}$

C. $\frac{1}{\lambda} \left(1 + \frac{1}{\lambda}\right)$

D. $\frac{1}{\lambda} \left(1 + \frac{1}{\lambda^2}\right)$

E. I don’t know
Solution

By the Feller paradox formula (7-32), she finds

\[ E(Z(t)) = \lambda E^0(T_1^2) \]

\[ = \lambda \times \left( \frac{1}{\lambda^2} + \sigma^2 \right) \]

where \( \sigma^2 \) is the variance of the exponential distribution

Now \( \sigma^2 = \frac{1}{\lambda^2} \)

Thus

\[ E(Z(t)) = \frac{2}{\lambda} \]

Answer B

For a Poisson process, the average duration of the time between events is \( 1/\lambda \), but the average duration of the current interval (seen by the inspector) is \( 2/\lambda \). This is Feller’s paradox. It is because an inspector is more likely to sample a long interval than a short one.
3. RWP and Freezing Simulations

Modulator Model:
We are given a sequence $Z_n$, $n \in \mathbb{Z}$, where $Z_n$ is called the modulator state at the $n$th epoch. We are also given a sequence $S_n > 0$, interpreted as the duration of the $n$th epoch. We assume that $(Z_n, S_n)$ is random, but stationary\(^1\) with respect to the index $n$. As usual, we do not assume any form of independence. We are interested in the modulated process $Z(t)$ defined by $Z(t) = Z_n$ whenever $t$ belongs to the $n$th epoch. We would like to apply Palm calculus to $Z(t)$. 

\(\text{State } Z(t)\)

\(Z_1\)

\(Z_2\)

\(Z_3 = Z(t)\)

\(Z_4\)

\(Z_5\)

\(Z_6\)

\(S_1\)

\(S_2\)

\(S_3\)

\(S_4\)

\(t\)

\(\text{Time } t\)
At $T_n$, channel state is drawn at random and new state is $i$ with proba $\pi_i^0$. When channel state $Z(t)$ is $i$, loss proba is $p_i$ and residence time in that state is (non random) $r_i$. What is the intensity of the point process $T_n$?

A. $\lambda = \sum_i \pi_i^0 r_i$

B. $\lambda = \sum_i \frac{\pi_i^0}{r_i}$

C. $\lambda = \frac{1}{\sum_i \pi_i^0 r_i}$

D. None of the above

E. I don’t know
Solution

By the intensity formula, $\lambda^{-1}$ is the mean duration of an interval, sampled at the beginning of an interval

$$\lambda^{-1} = E^0(T_1) = \sum_i \pi_i^0 r_i$$

Answer C
At $T_n$, channel state is drawn at random and new state is $i$ with proba $\pi^0(i)$. When channel state $Z(t)$ is $= i$, loss proba is $p_i$ and residence time in that state is $r_i$. What is the loss probability $p$ for a probe packet sent at an arbitrary point in time $t$?

A. $\frac{\sum_i \pi^0_i p_i r_i}{\sum_i \pi^0_i r_i}$

B. $\frac{\sum_i \pi^0_i r_i}{\sum_i \pi^0_i p_i r_i}$

C. $\frac{\sum_i \pi^0_i p_i r_i}{\sum_i \pi^0_i p_i}$

D. None of the above

E. I don't know
Solution

When the probe packet is sent, it finds the channel in some state $Z(t)$. The probability of loss, given the state $Z(t) = i$, is $p_i$. The loss probability, for the probe packet, is

$$\bar{p} = P(\text{loss}) = \sum_i P(\text{loss}|Z(t) = i)P(Z(t) = i) = \sum_i p_i P(Z(t) = i)$$

$$= E(p_{Z(t)})$$

By the inversion formula

$$E(p_{Z(t)}) = \lambda E^0 \left( \int_0^{T_1} p_{Z(s)} ds \right) = \lambda E^0 (p_{Z_1} T_1)$$

$$= \lambda \sum_i \pi_i^0 p_i r_i = \frac{\sum_i \pi_i^0 p_i r_i}{\sum_i \pi_i^0 r_i}$$

Answer A
Is the previous simulation stationary?

- Seems like a superfluous question, however there is a difference in viewpoint between the epoch $n$ and time.

- Let $S_n$ be the length of the $n^{th}$ epoch.

- If there is a stationary regime, then by the inversion formula

$$
\lambda = \frac{1}{\int_0^\infty t f_S(t) dt}
$$

so the mean of $S_n$ must be finite.

- This is in fact sufficient (and necessary).

**Theorem 7.9.** Assume that the sequence $S_n$ satisfies H1 and has finite expectation. There exists a stationary process $Z(t)$ and a stationary point process $T_n$ such that

1. $T_{n+1} - T_n = S_n$
2. $Z_n = Z(T_n)$
**Example 7.15: Random Waypoint, Continuation of Example 7.7.** For the random waypoint model, the sequence of modulator states is

\[ Z_n = (M_n, M_{n+1}, V_n) \]

and the duration of the \( n \)th epoch is

\[ S_n = \frac{d(M_n, M_{n+1})}{V_n} \]

where \( d(M_n, M_{n+1}) \) is the distance from \( M_n \) to \( M_{n+1} \).

Can this be assumed to come from a stationary process? We apply Theorem 7.9. The average epoch time is

\[ \mathbb{E}(S_0) = \mathbb{E}\left( \frac{d(M_n, M_{n+1})}{V_n} \right) = \mathbb{E}(d(M_n, M_{n+1})) \mathbb{E}\left( \frac{1}{V_n} \right) \]

since the waypoints and the speed are chosen independently. Thus we need that \( \mathbb{E}\left( \frac{1}{V_n} \right) < \infty \), i.e. \( v_{\text{min}} > 0 \).
Time Average Speed, Averaged over $n$ independent mobiles

- Blue line is one sample
- Red line is estimate of $\text{E}(V(t))$
A Random waypoint model that has no stationary regime!

- Assume that at trip transitions, node speed is sampled uniformly on \([v_{\text{min}}, v_{\text{max}}]\)
- Take \(v_{\text{min}} = 0\) and \(v_{\text{max}} > 0\)

- Mean trip duration = (mean trip distance) \( \times \frac{1}{v_{\text{max}}} \int_{0}^{v_{\text{max}}} \frac{dv}{v} = +\infty \)

- Mean trip duration is infinite!

- Was often used in practice

- Speed decay: “considered harmful” [YLN03]
What happens when the model does not have a stationary regime?

- The simulation becomes old
- When $\nu_{\text{min}} = 0$, sample average speed decays to 0

Model freezes

One User

Instant Speed + Empirical speed, both averaged over users
Stationary Distribution of Speed
(For model with stationary regime)

Random Waypoint on Rectangle, without Pause:

- Speed observed at waypoints \textbf{(Event average)}

- Speed observed at an arbitrary time \textbf{(Time average)}
Assume a stationary regime exists and simulation is run long enough

Apply inversion formula and obtain distribution of instantaneous speed $V(t)$

$$
\mathbb{E} (\phi(V(t))) = \lambda \mathbb{E}^0 \left( \int_0^{T_1} \phi(V(t)) \, dt \right)
$$

$$
= \lambda \mathbb{E}^0 (\phi(V_0) T_1)
$$

$$
= \lambda \mathbb{E}^0 \left( \phi(V_0) \frac{\|M_1 - M_0\|}{V_0} \right)
$$

$$
= \lambda \mathbb{E}^0 (\|M_1 - M_0\|) \mathbb{E}^0 \left( \frac{\phi(V_0)}{V_0} \right)
$$

$$
= C \int_0^{v_{\text{max}}} \frac{\phi(v)}{v} f_{V_0}(v) \, dv
$$

$$
f_{V(t)}(v) \, dv = \frac{C}{v} f_{V_0}'(v) \, dv$$
A (true) example: Compare impact of mobility on a protocol: Experimenter places nodes uniformly for static case, according to random waypoint for mobile case. Finds that static is better

Find the bug!

A. Spatial distribution of nodes is different for mobile case than for static case
B. Spatial distribution of nodes is same for mobile and static cases but speed is more often small than large
C. There is no bug, mobility increases capacity
D. I don’t know
Solution

In the mobile case, the nodes are more often towards the center, distance between nodes is shorter, performance is better.

The comparison is **flawed**. Should use for static case the same distribution of node location as random waypoint.

*Is there such a distribution to compare against?*
A Fair Comparison

Revisit the comparison by sampling the static case from the stationary regime of the random waypoint.
A mobile moves as follows
- pick a random direction uniformly in $[0, 2\pi]$
- pick a random trip duration $T \sim \text{Pareto}(p)$
- go in this direction for duration $T$ at constant speed; if needed reflect at the boundary.

A. Yes if $p > 1$
B. Yes if $p > 2$
C. Yes for all $p$
D. No
E. I don’t know
The only condition is that mean trip duration be finite. The trip duration is Pareto($p$). It has a finite mean iff $p > 1$.

Answer A
4. PASTA

- There is an important case where Event average = Time average
- “Poisson Arrivals See Time Averages”
  - More exactly, should be: Poisson Arrivals independent of simulation state
    See Time Averages

Consider a system that can be modeled by a stationary Markov chain $S(t)$ in discrete or continuous time (in practice any simulation that has a stationary regime and is run long enough). We are interested in a matrix of $C \geq 0$ of selected transitions such that

**Independence** For any state $i$ of $S(t)$, $\sum_j C_{i,j} \overset{\text{def}}{=} \lambda$ is independent of $i$.

**Theorem 7.13 (PASTA).** Consider a point process of selected transitions as defined above. The Palm probability just before a transition is the stationary probability.

- E.g. an observer that arrives according to a Poisson process of any rate, independent of system state, see the stationary distribution (= at an arbitrary point in time)
Consider an M/M/1 queue with $\rho < 1$. Does the point process of departures satisfy PASTA?

A. No
B. Yes
C. It depends on the parameters
D. I don’t know
Solution

The state just before a departure satisfies $N \geq 1$ where $N$ is the number of customers, therefore we never see $N = 0$ just before a departure.

In contrast, for an observer, i.e. under the stationary probability, we see $N = 0$ with probability $1 - \rho > 0$

PASTA does not hold, answer A
Solution

We can also see that the assumptions of PASTA do not hold:

\[ \text{N}(t) \text{ is Markov} \]

\[ \text{if } N(t) > 1 \text{ then proba of a departure in } [t, t + dt] = \mu dt + o(dt) \]

\[ \text{else if } N(t) = 0 \text{ then proba } = 0 \]

\[ \Rightarrow \text{ NOT PASTA - able} \]
Consider an M/M/1 queue with $\rho < 1$. Does the point process of arrivals satisfy PASTA?

A. No  
B. Yes  
C. It depends on the parameters  
D. I don’t know
Solution

The arrival process is Poisson and independent of system state, therefore PASTA holds. Answer B
In an M/M/1/K queue, the probability that an arriving job is discarded is...

A. $P(N = K)$
B. $P(N = K - 1)$
C. $P(N = K + 1)$
D. None of the above
E. I don’t know
Solution

By the PASTA property, an arriving customer sees the stationary distribution

\[ \begin{align*}
\mathbb{P}(N = k) &= \eta (1 - \rho) \rho^k 1_{\{0 \leq k \leq K\}} \\
\eta &= \frac{1}{1 - \rho^{K+1}} \\
\mathbb{P}^0(\text{arriving customer is discarded}) &= \mathbb{P}(N = K)
\end{align*} \]
Example 11.15: A Poisson Process that does not Satisfy PASTA. The PASTA theorem requires the event process to be Poisson or Bernoulli and independence on the current state. Here is an example of Poisson process that does not satisfy this assumption, and does not have the PASTA property.

Construct a simulation as follows. Requests arrive as a Poisson process of rate \( \lambda \) into a single server queue. The service time of the request that arrives at time \( T_n \) is \( \frac{1}{2}(T_{n+1} - T_n) \). The service times are exponential with mean \( \frac{1}{2\lambda} \), but not independent of the arrival process. The system has exactly one customer during half of the time, and 0 customer otherwise. Thus the stationary distribution of queue length \( X_t \) is given by \( \mathbb{P}(X_t = 0) = \mathbb{P}(X_t = 1) = 0.5 \) and \( \mathbb{P}(X_t = k) = 0 \) for \( k \geq 2 \). In contrast, the queue is always empty when a customer arrives. Thus the Palm distribution of queue length just before an arrival is different from the stationary distribution of queue length.

The arrival process does not satisfy the independence assumption: at a time \( t \) where the queue is not empty, we know that there cannot be an arrival; thus the probability that an arrival occurs during a short time slot depends on the global state of the system.
5. Perfect Simulation

- An alternative to removing transients
- Possible when inversion formula is tractable
- Example: random waypoint
  - Same applies to a large class of mobility models
Perfect Simulation

- Def: a simulation that starts with a sample from the stationary distribution
- Usually difficult except for specific models, e.g. Random waypoint:
  - Sample Prev and Next waypoints from their joint stationary distribution
  - Sample M uniformly on segment [Prev, Next]
  - Sample speed from its stationary distribution
Stationary Distrib of Prev and Next

- Let $M(t)$: position at time $t$
- Let $Prev(t), Next(t)$: previous and next waypoints
Is $M(t)$ uniformly distributed?

A. Yes
B. No
C. It depends on the distribution of speed
D. I don’t know
Is $Next(t)$ uniformly distributed?

A. Yes
B. No
C. It depends on the distribution of speed
D. I don’t know
Solution

The distribution of $M(t)$ is not uniform (more towards the center)

The distribution of Next($t$) is not uniform (more towards the corners)

The distribution of Prev($t$) is not uniform (more towards the corners)

The distributions of Next($t$) and Prev($t$) are the same
Stationary Distribution of Location Is also Obtained By Inversion Formula

Joint distribution of \((Prev(t), M(t), Next(t))\) has a simple closed form [NavidiCamp04]:

1. \(((Prev(t), Next(t)))\) has density over area \(A\)

\[ f_{Prev(t),Next(t)}(P, N) = K \|P - N\| \]

2. Distribution of \(M(t)\) given \(Prev(t) = P, Next(t) = N\) is uniform on segment \([P, N]\)

\[ K^{-1} = \text{vol}(A)^2 \bar{\Delta}(A), \text{ with } \bar{\Delta}(A) = \text{average distance between two points in } A. \text{ For } A = [0; a] \times [0; a], \bar{\Delta}(A) = 0.5214a \text{ [Gosh1951].} \]
Proof. For any bounded, non-negative function $\phi$:

$$
\mathbb{E}(\phi(\text{Prev}(t), M(t), \text{Next}(t))) = \lambda \mathbb{E}^0 \left( \int_0^{T_1} \phi \left( M_0, M_0 + \frac{t}{T_1} (M_1 - M_0), M_1 \right) dt \right).
$$

By a simple change of variable in the integral, we obtain

$$
\lambda \mathbb{E}^0 \left( T_1 \int_0^1 \phi(M_0, M_0 + u(M_1 - M_0), M_1) du \right).
$$

Now given that there is an arrival at time 0, $T_1 = \frac{\|M_1 - M_0\|}{V_0}$ and the speed $V_0$ is independent of the waypoints $M_0$ and $M_1$ thus

$$
= \lambda \mathbb{E}^0 \left( \frac{1}{V_0} \right) \mathbb{E}^0 \left( \|M_1 - M_0\| \int_0^1 \phi(M_0, M_0 + u(M_1 - M_0), M_1) du \right)
$$

$$
= K_2 \int_A \int_A \int_0^1 \phi(M_0, (1 - u)M_0 + uM_1, M_1) \|M_1 - M_0\| \ du \ dM_0 \ dM_1
$$

which shows the statement.  \[\square\]
No Speed Decay

Standard Simulation

Perfect Simulation
Perfect Simulation of RWP: How do you sample the speed?

A. By rejection sampling
B. By CDF inversion
C. By an ad-hoc method
D. I don’t know

\[ f_{V(t)}(v) dv = \frac{C}{v} f_{V_0}^0(v) dv \]
Perfect Simulation of RWP: How do you sample the speed?

1. \([(\text{Prev}(t), \text{Next}(t))]\) has density over area A

\[ f_{\text{Prev}(t),\text{Next}(t)}(P, N) = K \| P - N \| \]

A. By rejection sampling
B. By CDF inversion
C. By an ad-hoc method
D. I don’t know
Perfect Simulation Algorithm

- Sample a speed $V(t)$ from the time stationary distribution by CDF inversion
- Sample $\text{Prev}(t)$, $\text{Next}(t)$ by rejection sampling
- Sample $M(t)$ uniform on $[\text{Prev}(t) ; \text{Next}(t)]$

1. Draw $(M_0, M_1)$ with joint density $K \|M_1 - M_0\|$ on $A \times A$: do
   - draw $M_0, M_1 \sim \text{Unif}(A)$
   - draw $V \sim \text{Unif}[0, \Delta]$
   - until $V < \|M_1 - M_0\|$  

2. Draw $U \sim \text{Unif}[0, 1]$

3. $M(t) = (1 - U)M_0 + UM_1$

$\Delta$: upper bound on diameter $\bar{\Delta}$ of area $A$
Conclusions

- A metric should specify the sampling method
- Different sampling methods may give very different values
- Palm calculus contains a few important formulas
- Freezing simulations are a pattern to be aware of